Yablo's Series Numeric Versions

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Abstract

At first glance, the most significant element in debates about Yablo's paradox has been the problem of circularity. However, on closer investigation, a second level controversy emerges, that is, the selection of a formally correct language in which the sequence can be formulated, and by following deductive inference, a contradiction is derived. In fact, this latter element is the most important–the existence of Yablo's endless sequence. What kind of entities are we talking about? Is it a list of formulae without interpretation? Or a sequence of statements, where statements are the truth-bearers? I will demonstrate that the Yablo's paradox may be formulated without the use of semantic predicates, sentence variables, and the quotation marks it implies.

The three points of view are important to understand the Yablo's paradox. Let us take them in turn.

1 The paradox can be formulated without the use of the truth predicate

The original series looked like this:

(S_1) for all $k > 1, S_k$ is untrue
(S_2) for all $k > 2, S_k$ is untrue
(S_3) for all $k > 3, S_k$ is untrue
$\ldots$

In the sequence, the dots being referred to be continuous also occurred in the original article.[1] Note the point where the names of the formulas, which are the symbols in the left brackets, are themselves part of the formulas. Let us then reformulate the paradox. Let n, k be natural numbers ($n = 0, 1, 2 \ldots$), and then we can formulate the Yablo's series of sentences as follows:

$$
\begin{array}{rl}
(Y_0) & S(0) \leftrightarrow \forall k.0 < k \rightarrow \neg S(k) \\
(Y_1) & S(1) \leftrightarrow \forall k.1 < k \rightarrow \neg S(k) \\
(Y_2) & S(2) \leftrightarrow \forall k.2 < k \rightarrow \neg S(k) \\
 & \ldots \\
(Y_n) & S(n) \leftrightarrow \forall k.n < k \rightarrow \neg S(k) \\
(Y_n + 1) & S(n+1) \leftrightarrow \forall k.n+1 < k \rightarrow \neg S(k)
\end{array}
$$

Now, the names of the formulas are not part of the formulas, and the formulas are first-order and do not use the semantic predicate, "true." It can be observed that the sequence $S(n)$ of the Yablo's paradox is a consequence of a second-order logical formula below:

$$(Ya) \quad \exists s \forall n : n \text{ natural number} \rightarrow (s(n) \leftrightarrow \forall k(n < k \rightarrow \neg s(k)))$$

It is provable, that Ya is false.

2 The paradox is apparently related only to logic, its foundations

In fact, the Yablo's paradox can be formulated without logical concepts (e.g., with numbers). Let the usual arithmetic truths (AR) hold, and the value of the binary function f is 1 for some number n, if and only if for every number greater than n, the value of f is 0 as follows:

[1]Yablo, Stephen (1993) "Paradox without Self-Reference" *Analysis* 53: 251–52. `http://www.mit.edu/~yablo/pwsr.pdf`.

$$
\begin{array}{rl}
(Y0) & 1 = f(0) \leftrightarrow \forall k.0 < k \rightarrow 0 = f(k) \\
(Y1) & 1 = f(1) \leftrightarrow \forall k.1 < k \rightarrow 0 = f(k) \\
(Y2) & 1 = f(2) \leftrightarrow \forall k.2 < k \rightarrow 0 = f(k) \\
 & \ldots \\
(Yn) & 1 = f(n) \leftrightarrow \forall k.n < k \rightarrow 0 = f(k) \\
(Yn+1) & 1 = f(n+1) \leftrightarrow \forall k.n+1 < k \rightarrow 0 = f(k)
\end{array}
$$

The antinomy can then be derived as follows:

$\star(1)$	$1 = f(n)$	Assume that the value of f for n is 1
$\star(2)$	$\forall k.n < k \rightarrow 0 = f(k)$	(Yn)
$\star(3)$	$n < n+1 \rightarrow 0 = f(n+1)$	(2)
$\star(4)$	$n < n+1$	(AR)
$\star(5)$	$0 = f(n+1)$	(3) (4)
$\star(6)$	$(\forall k.n < k \rightarrow 0 = f(k)) \rightarrow (\forall k.n+1 < k \rightarrow 0 = f(k))$	(AR)
$\star(7)$	$\forall k.n+1 < k \rightarrow 0 = f(k)$	(2) (6)
$\star(8)$	$1 = f(n+1)$	(7) $(Yn+1)$
(9)	$1 = f(n) \rightarrow .0 = f(n+1)\ \&\ 1 = f(n+1)$	(1) (5) (8)
(10)	$0 = f(n)$	(9)
(11)	$\forall n.0 = f(n)$	(10) n was arbitrary, so by universal generalization
(12)	$\forall k.n < k \rightarrow 0 = f(k)$	(11) (AR)
(13)	$1 = f(n)$	(Yn)
(14)	$0 = f(n)\ \&\ 1 = f(n)$	(10) (13)

The inconsistency disappears if we add a transfinite number (e.g. ω) to the domain of the sequence, which is a transfinite number greater than any natural number.

Suppose $1 = f_2(\omega)$ and $1 = f_2(n) \leftrightarrow \forall k.n < k \rightarrow 0 = f_2(k)$.

The finite segment of the sequence can be written in numerical forms so that it can be calculated using a spreadsheet. In this case, instead of a universal quantifier, we can use a sum of many cells, but other solutions are also possible.

Indeed, infinite sums of many cells cannot be represented by a finite spreadsheet, but we can simulate the paradox by representing the serial numbers in a circular manner. In this approach $\omega + 1 = 1$ and the value of the

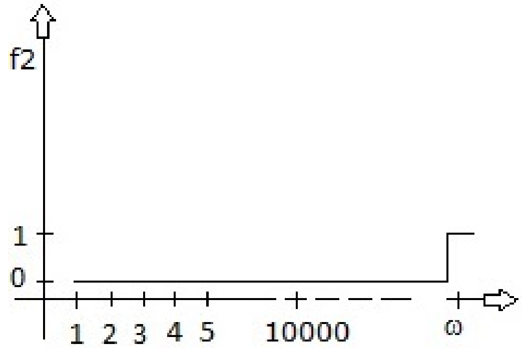

Figure 1: f_2 function

last cell will be determined by the value of the first cell. The spreadsheet works according to the following formula below:[2]

$$1 = f_2(\omega) \text{ és } 1 = f_2(n) \leftrightarrow 0 = \sum_{k=n+1}^{k=\omega} f_2(k)$$

Without specifying the base of the series, the numeric formula for the paradox would be as follows:

$$1 = f_3(n) \leftrightarrow 0 = \sum_{k=n+1}^{k=\infty} f_3(k)$$

The point is that there is no such f_3 function. The Yablo's paradox is about the nature of sequences or functions.

3 What does the paradox mean in this case?

It means that we have assumed the existence of something, although this assumption leads to a contradiction. What does this imply? It follows that we must reject our assumption that the sequence we assumed to exist does not in fact exist. It is important to understand this as well. We are not denying the existence of a sequence of signs because this paper can take anything, although the existence of a given sequence of logical formulas suggests that each member can be consistently evaluated to be true or false. The so-called paradox arises from the fact that we have not given the basis for a

[2]See `https://ferenc.andrasek.hu/models/ybrx-brief.xlsx`

particular recursive sequence. Once given, the contradiction and the paradox disappears, despite the infinite members of the sequence. Because the base of a recursive sequence is missing, the definition of the sequence is circular, that is, incorrect (Stephen Yablo does not think that circular definitions are necessarily wrong; he has his own interesting theory, but I cannot go into it here).