On Russell's Paradox with Nails and Strings

Ferenc András
ORCID iD: 0009-0002-9995-8173

1985-2025

Keywords: paradox, Russell's paradox

Abstract

Russell's paradox challenges the foundations of naive set theory, questioning the assumption that every predicate determines a corresponding set. This paper explores the paradox through an alternative illustrative framework using nails and strings, offering an intuitive yet rigorous representation of the membership relation. By analyzing how self-referential structures emerge within this model, the paper highlights three possible resolutions: hierarchical, restrictive, and temporal. The discussion ultimately suggests that a weakened, nominalist interpretation of set formation can preserve the principle Russell refuted, albeit with significant conceptual modifications.

1 Introduction

Russell's paradox concerns the foundations of naive set theory. This short paper explores how it can be interpreted in other contexts and its significance in the world of automata. The discussion assumes that the reader is broadly familiar with the foundations of set theory and its historical development. Since the text contains numerous formulas, the reader should also be comfortable working with logical expressions.

2 Russell's Paradox and Its Formal Presentation

In 1901, Bertrand Russell refuted the following assumption: (A) for every one-place predicate α, there exists a set whose members are precisely those elements for which α holds true. To refute this assumption, he employed the predicate 'not a member of itself.' Quine wrote in one of his textbooks:

> "As soon as we waive his type distinctions and read (A) simply as:[1]

(A') $\exists y \forall x (x \in y \leftrightarrow \alpha(x))$,

we find ourselves in trouble. For, introducing '¬①∈①' at the occurence of 'α', we can deduce a palpable falsehood as follows:

(1)	$\exists y \forall x [x \in y \leftrightarrow \neg(x \in x)]$	
(2)	$\forall x [x \in y \leftrightarrow \neg(x \in x)]$	(1) y
(3)	$y \in y \leftrightarrow \neg(y \in y)$	(2)
(4)	$\exists y [y \in y \leftrightarrow \neg(y \in y)]$	(3)

This difficulty is called *Russell's paradox*, for its discoverer (1901)."

3 An Alternative Illustration: Nails and Strings

To fully grasp the assumption and the essence of the proof, I present a straightforward example. I have slightly modified the original proof while preserving its core idea.

Sets (or hypersets) can be explained in different ways depending on the purpose of the explanation. Some formalizations of set theory can be illustrated using plane figures. However, I reject this approach here, as it does not effectively demonstrate cases where a set is a member of another set or itself. Instead, I propose an alternative illustration that provides deeper insight into the problem.

Consider a room with steel nails driven into the floor. Some of these nails are connected as follows: two nails can be connected in at most one direction, and any string connecting them has a distinct beginning and end. (Figure 1) The arrows represent the membership relation. From the Figure

[1] Quine, Willard Van Orman. *Methods of Logic*. London: Routledge & Kegan Paul (1958): 249.

1, we can observe the following sets: $A = \{A\}; B = \{C, D\}; C = \{C\}; F = \{D, G, F\}; H = \{D, G\}$.

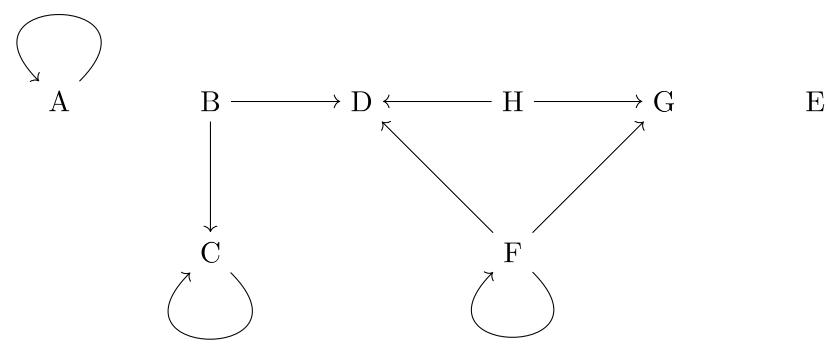

Figure 1:

4 Possible Resolutions to the Paradox

Some nails are tied with a loop, while others are not. Our task is as follows: we are given a silver nail and a brass nail. The silver nail must be connected to every nail in the room that has no loop around it, with the connection starting from the silver nail. The brass nail must be connected to every nail that has a loop around it, with the connection starting from the brass nail. The task is only complete if we find a unique and correct solution. Figure 2 illustrates the room again, now including the two new nails.

The silver nail does not need to be connected to nails g, d, and e, as they do not have strings starting from them. Nor should it be connected to nails a, c, and f, since they have loops around them. However, it must be connected to nails b and h, as these have strings starting from them but no loops. This leads to the crucial question: should the silver nail be connected to itself? Since it has a string starting from it towards b and h and has no loop, it should be. However, as soon as we do so, we create a contradiction: it now has a loop, meaning it should not be connected. Thus, the task for the silver nail is unsolvable.

For the brass nail, we find two possible solutions, which is also problematic, as it leaves us uncertain about which one to choose.

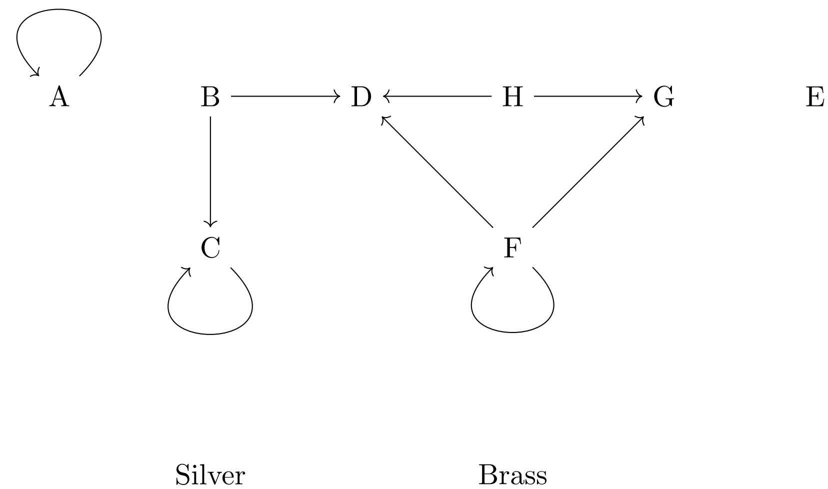

Figure 2:

If we interpret the nails as sets (or hypersets) and the strings as the membership relation, we can express the task using the following two interpreted formulas:

(1)	$\forall x : x \in \text{silver} \leftrightarrow (\exists y\, y \in x \& x \notin x)$	
	$\forall x : x \in \text{brass} \leftrightarrow (\exists y\, y \in x \& x \in x)$	
	The task is unsolvable for the silver nail because (1) leads to a contradiction:	
(2)	$\text{silver} \in \text{silver} \leftrightarrow (\exists y\, y \in \text{silver} \& \text{silver} \notin \text{silver})$	(1)
(3)	$b \notin b \& d \in b)$	Figure 1
(4)	$\exists y (y \in b \& b \notin b)$	(3)
(5)	$b \in \text{silver}$	(1)(4)
(6)	$\exists y (y \in \text{silver})$	(5)
(7)	$\exists y (y \in \text{silver}) \rightarrow (\text{silver} \in \text{silver} \leftrightarrow \text{silver} \notin \text{silver})$	(2)
(8)	$\text{silver} \in \text{silver} \leftrightarrow \text{silver} \notin \text{silver}$	(6)(7)

If there were no steel nails in the room, the task for the silver nail could be solved as follows: $\neg \exists y (y \in \text{silver})$. However, even in this case, the definition of the brass nail—and its corresponding set—remains ambiguous, as a set is uniquely defined by its elements. In this scenario, neither the statement ‘brass $\in$ brass’ nor ‘brass $\notin$ brass’ contradicts assumption (1). There are three possible ways to resolve this paradox:

4.1 I. The Hierarchical Approach

The silver and brass nails must be placed outside the room. According to Russell, they should be positioned on the floor above, whereas Zermelo suggests that placing them in the hallway is sufficient. In this case, the solution is identical to what it would be if definition (1) applied only to the steel nails. If we take the steel nails to represent sets, the necessity of leaving the room becomes clear. Let U be the set of nails in the room:

$$\begin{array}{lll} (9) & \forall x(x \in \text{silver} \leftrightarrow (x \in U \& \exists y\,(y \in x) \& x \notin x)) & \\ (10) & \text{silver} \in \text{silver} \leftrightarrow (\text{silver} \in U \& \exists y(y \in \text{silver}) \& \text{silver} \notin \text{silver}) & (9) \\ (11) & \text{silver} \in U(\text{silver} \in \text{silver} \leftrightarrow \text{silver} \notin \text{silver}) & (10)(6) \\ (12) & \text{silver} \notin U & (11) \end{array}$$

Thus, neither the silver nail nor the brass nail will have a loop around it.

4.2 II. The Restriction Approach

We can impose a restriction that the silver and brass nails must not be tied with a loop. However, this solution cannot be applied to sets. Quine pointed out that the axiom equivalent to this solution (13) leads to a contradiction.[2]

$$(13)\ \forall F \exists y \forall x(y \notin y \& (x \neq y \rightarrow (F(x) \leftrightarrow x \in y)))$$

4.3 The Temporal Approach

In this approach, the nails are not considered sets but rather symbols, and definitions are understood as actions occurring over time. Introducing a definition means issuing instructions, while applying a definition means following those instructions. The paradox arises because the nails that must be connected to the new ones are not initially defined by a list. If there are too many nails, listing them individually is impractical or even impossible. Instead, we use definitions or rules. These rules must be applied to objects

[2] Quine, W.V. "On Frege's Way Out." *Mind* Vol. 64, No. 254 (Apr. 1955): 145-159.

that are fully determined at the outset. Thus, the task's solution cannot be applied to itself before the task is completed. In other words, if a predicate of an object depends on the solution, then this predicate cannot be part of the initial conditions. To resolve this, we must take a "photograph" of the room's original state, before adding the silver and brass nails, and verify the solution afterwards.

Since the nails serve as an illustration of sets, the following axioms concerning time and symbol usage must be fulfilled:

(14) Time is an infinite sequence of discrete moments.

(15) For every γ predicate, at any time t_1, there is a symbol, and at the following moment in time t_2, the symbol at t_2 denotes those and only those things that have γ predicate at t_1.
e.g. $\gamma(x) := \exists y(y \in_1^\star x)$ & $x \notin_1^\star x$ where:
$y \in_1^\star x := y$ is element of x at t_1
$y \in_2^\star x := y$ is element of x at t_2
$y \in_3^\star x := y$ is element of x at t_3
...
$x \in_2^\star$ silver $\leftrightarrow \exists y(y \in_1^\star x)$ & $x \notin_1^\star x$
(The temporal ordering of the modified membership relations is similar to Russell's type theory.)

(16) Sets are symbols that denote or do not denote things, irrespective of time.

5 Conclusion

Under this weakened, nominalist conception, assumption (A), refuted by Russell, remains valid.

Email: ferenc@andrasek.hu