On the arrow paradox

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Abstract

Many philosophers struggle to understand how Newtonian physics and mathematical analysis address Zeno's arrow paradox. They argue that if a physical object occupies a precisely defined location at a specific instant in time, it must be stationary and not in motion. Consequently, they contend that motion cannot be understood in such terms. In this paper, I will demonstrate, through various explanations and illustrative examples, that this perspective is incorrect. I will show that an object can indeed be in motion at a single instant in time.

Preface

Nicholas Fearn's witty and well-written book *Zeno and the Tortoise: How to Think Like a Philosopher* has made philosophical thinking accessible to many readers (Fearn 2011). However, his presentation of the well-known Zeno's Arrow paradox is less successful. In what follows, I will focus exclusively on this chapter of his book.

Understanding the paradox in Fearn's approach

> The flight of an arrow can be divided into instants, which are the smallest possible measure of time. If the arrow moves during one of these instants, it means that it begins the instant in one place and ends in another. In this case we would not be talking about an instant at all because the instant could be divided further. Once we have alighted on a true instant—a moment that by definition cannot be divided further—then we have a division of time in which no movement can take place. This,

> however, means that the arrow can never move, as no amount of no-motion can add up to motion. Since the arrow does not move in any single point in its flight, it does not move over the whole flight. The arrow is the easiest of the paradoxes to tackle. Motion requires time, so it is not surprising that if you take away time and talk instead of instants then you also take away motion. Though the arrow may not move in any given instant, it can still move if motion is defined as a thing's appearance in a different place at a later point in time.[1]

The reasoning above is premature. It conflates two issues: confusing what we know with what exists. In this context, there are two distinct questions:

(1) What can we know when we consider a moment, a fraction of time that cannot be divided any further? Can we determine whether the arrow is stationary or moving in such a moment?

(2) Can an object—an arrow in this case—move or remain still within an indivisible fraction of time? Note that standing still is as much a question as moving. Consider a bouncing ball: the ball is in constant motion but periodically comes to rest for a moment when it bounces off the ground.

Returning to the quote, let us examine Fearn's argument more closely. Let's start at the end. If the arrow does not move at any instant of time, why and how does it appear to move? Moreover, how does this resolve the problem of motion? How should we understand the concept of appearance in this case?

Appearances are typically understood in contrast to reality. For example, a stick submerged in water appears bent, but this is an optical illusion—the stick is actually straight. A magician appears to saw a woman in half, but in reality, he does not; the woman smiles and steps out of the box unharmed. Similarly, the arrow appears to move, but if it is in the same place all the time, it is not moving in reality.

This might occur if, for instance, the arrow is lying on a table while we are sitting in a moving car. From our perspective, it might appear as though the arrow is moving, even though it remains stationary relative to the table. However, does this solve the paradox of the moving

[1] Nicholas Fearn, *Zeno and the Tortoise: How to Think Like a Philosopher* (2001) London, Atlantic Books. pp.82,83. At the time of writing Nicholas Fearn was a student at King's College, London and living in London. Not to be confused with the journalist.

arrow? Not entirely—it merely shifts the problem elsewhere. In this case, we would now need to explain the motion of the car instead. Thus, the fundamental problem remains unresolved.

It seems that Zeno may indeed have identified a contradiction in the concept of motion. Let us delve deeper into this issue.

An Approach to the Arrow Aporia

I will explain this paradox using natural language, as well as a more precise, semi-formal language. The numbers in brackets on the right indicate what each statement follows from.
Let us begin by assuming that the arrow moves, which I denote as (1). The asterisk indicates that this is an assumption rather than a logical truth. Further assumptions will also be marked with additional asterisks. The numbers in parentheses on the right indicate what each statement is derived from.

(1)	The arrow moves.	
(2)	The arrow is at a specific position at every given moment, and each part of it fully occupies the space available to it.	
(3)	The arrow is where it is at all times, moving nowhere within a moment. It cannot move in the direction of the position it occupies because it is already there, nor can it move away from that position because there is no time for such motion in a moment.	(2)
(4)	The arrow is not moving at any given moment, so it is not moving at any moment.	(3)
(5)	The arrow does not move.	(4)

This reasoning produces an obvious contradiction between (1) and (5). Let us now examine the logic in greater detail:

*(1)	The arrow moves at time t_1.	
**(2)	The arrow is at a specific position s on the track at each time instant t.	
**(3)	The arrow is at position s1 at time t_1.	(2)
***(4)	If the arrow moves, it must exist in a time domain.	
***(5)	If the arrow does not exist in a time domain, it cannot move.	(4)
****(6)	An instant in time is never a time domain.	
****(7)	The time instant t_1 is not a time domain.	(6)
****(8)	The arrow does not exist in a time domain at t_1.	(7)
****(9)	The arrow does not move at t_1.	(5)(8)
****(10)	The arrow moves at t_1, and the arrow does not move at t_1.	(1)(9)

Since t_1 was arbitrary, this contradiction applies to any given moment in time.

Resolving the Contradiction

Given that the reasoning is valid, one or more of the initial premises must be incorrect. Which one should we discard? Let us review the key premises (marked with asterisks), now labelled with Roman numerals:

(I) The arrow moves at time t.

(II) The arrow is at a defined position s on the track at each instant t.

(III) If the arrow moves, it must do so within a time domain.

(IV) A moment in time is never a time domain.

Premise (IV) appears irrefutable. Why? Because an instant of time is indivisible, whereas a time domain (assuming continuous time) can always be subdivided further. Something is either a time instant or a time domain, but this distinction does not preclude the possibility that a time domain consists of time instants. Assuming discrete time, the shortest time span would consist of two consecutive instants. However, in continuous time, the concept of the "next instant" after a point in time becomes meaningless. (Do not confuse the concept of a continuum with continuity; the latter can apply in both discrete and continuous time frameworks.)
Premise (II) also seems unassailable. If the trajectory of the arrow is described by a function s, this function assigns a unique value to each time t. Therefore, for every time t, there is a corresponding position $s(t)$.

Thus, we are left with a choice:

- Either we accept (I), in which case Zeno is correct.
- Or we accept (III), aligning with Newtonian mechanics.

Examining Premise (III)

Premise (III) states that if the arrow is at position s_1 at time t_1, it is stationary. Before analysing this mathematically, consider an illustrative example. Below are two diagrams (Figure 1). One depicts an arrow stationary at t_1, while the other depicts an arrow moving at t_1. Can we determine which arrow is moving—the one on the left or the one on the right? We cannot

Figure 1: Two arrows

tell, as we lack any reference to discern motion. The problem is this: an arrow cannot move at all within a single moment. Yet this does not imply that the arrow's velocity is zero at that moment. While counterintuitive, this statement is true.

Next Steps

How can we demonstrate this? I will explain it using two approaches: one based on elementary principles and another assuming knowledge of secondary school mathematics.

What Did We Learn in Secondary School?

The elementary school curriculum teaches that distance = speed × time. This is not entirely accurate, but more precisely: the distance traveled by the arrow = the speed of the arrow × the time taken. In formula: $s = v \times t$. Zeno deduced from the fact that $s = 0$ that $v = 0$, i.e., the arrow is stationary. However, this is a wrong conclusion, because it is only one possibility; the other possibility is that the time elapsed, i.e., $t = 0$, and the velocity is not zero. His conclusion was therefore wrong, but Zeno did not know such formulas or the concept of velocity, so he cannot be blamed for the error.

At the secondary level, we can be more precise and general. The position of the arrow at any instant in time is given by the travel-time function s over a time interval T, but what describes its speed? Assuming that the travel-time function of the arrow is differentiable at each instant of time T, the velocity of the arrow is described by the derivative function s' of the function s in time T. The derivative function then gives the velocity of the arrow at each time instant, at least according to Newtonian physics. So, it is indeed possible to interpret the speed of the arrow at all time instants, not just in one time domain. The arrow has a velocity at every time instant, and it is stationary at some time instant t if the value of the velocity function at that time instant is zero, moving otherwise. For example, when the arrow bounces, it stops for a moment, similar to a bouncing ball. However, if we only know the position of the arrow at a single point in time, we cannot determine the function describing its trajectory, and hence the velocity function. If we know only a single time-place value, we cannot determine the differential quotient of the function at that point, because we cannot determine the tangent to that function value. If the tangent cannot be determined, then the velocity cannot be determined either, i.e., we do not know whether the arrow is stationary or moving. Therefore, we cannot choose between the two figures above (1.

Let's recall high school physics class. Suppose the arrow is flying along a straight line with a uniform speed v in time t. Then its trajectory is described by the formula $s = v \times t$, where s is the distance travelled, v is the velocity, and t is the elapsed time. The velocity-time function is a constant function, as shown in the figure. In high school, this was also represented by a graph. Observe carefully Figure 2.

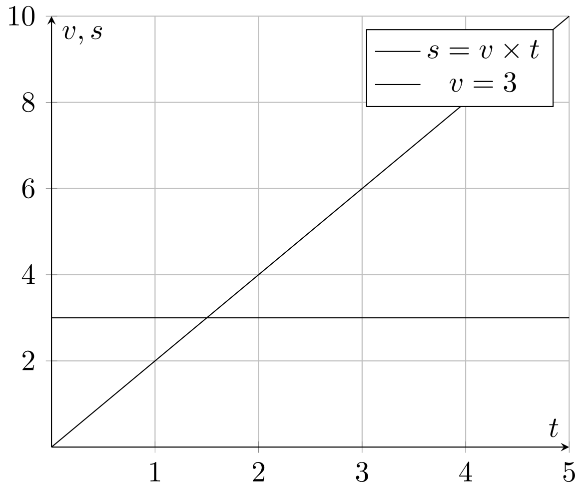

Figure 2: Flying arrow

The figure 2. shows that the arrow is exactly where it is at each instant of time, and that the arrow has a velocity $0 < v$ at each instant of time, so it is not stationary but moving along. The fact that it is moving all the time, i.e., the velocity-time function is continuous, means that it is moving at each time instant, even though it does not move at any particular time instant. This is still incomprehensible to many people today.

Richard Mark Sainsbury, in his popular book on paradoxes, also mentions the arrow paradox (Sainsbury 2012). He presents a solution to the problem in the spirit of Newton, but for him, it is not "the solution", but only one of the possible solutions, i.e., just an opinion. I believe he is wrong; it is impossible to evade the solution of classical physics on this issue. Let us consider the following, sticking with the example of the arrow. The numbers in parentheses on the right show what each line follows from, with the reasoning behind it. The example can be generalized:

*(1)	The arrow is somewhere at every time t.	
*(2)	The arrow has position $s(t)$ at each time instant t.	(1) common sense
*(3)	There exists a path-time function of the arrow s.	(2) mathematics
**(4)	Assume that the path-time function of the arrow is differentiable at each time instant. This second premise is the assumption of classical physics.	
**(5)	There exists a derivative function s' describing the speed of the arrow.	(4) mathematics
**(6)	At every time instant t, the arrow has a fixed velocity $s'(t)$, which may or may not be zero; that is, at every time instant, the velocity of the arrow exists.	(5) physics
(7)	If the arrow is somewhere at every time instant t, and the path-time function is derivable, then the fact that at some time instant t_1 there is a location such that $a = s(t_1)$ does not imply that $0 = s'(t_1)$.	

The solution to Newtonian physics is only arguable if it is arguable that the arrow is somewhere at every time instant t, or that the path-time function is differentiable. On the latter question, physics is competent; philosophy's competence is limited to the question of whether the arrow is somewhere at every time instant t. Is the latter position questionable? I think not, and therefore the solution of classical physics is not questionable.

It follows from this that the arrow paradox does not prove the contradictory nature of motion, and provided that the scope of contradictory concepts is empty, it does not disprove the existence of motion. If there is any mystery about motion, it is why, more than three hundred years after Newton, so many contemporary philosophers keep repeating it.

Addendum to the arrow paradox

Is there empirical evidence that the arrow has a speed at each instant of time? Yes, there is. Place a tiny magnet on the arrowhead, move it through a solenoid of sufficiently large diameter, and observe the induced voltage curve on an oscilloscope. As the arrow moves inside the solenoid, a voltage will be continuously measured, due to the magnet moving with the arrow. This proves that the moving arrow has a velocity at each instant of time.

What would be the consequences of assuming an atomic structure of time and space? The notion of distance could not be unambiguously defined in the usual way in two- or higher-dimensional space. In the case of a change in velocity at discrete time intervals, a different velocity would result from the previous and the next "time atom", i.e., the velocity would not be unambiguous with respect to the instantaneous position of the body. Perhaps both problems can be remedied in some way, but the very occurrence of this issue clearly demonstrates how justified and well-founded the classical approach is.

Bibliography

- Boccardi, Emiliano (2019) Change and Contradiction: A Criticism of the Hegelian Account of Motion, in Rodrigo Freire Edgar Almeida & Alexandre Costa-Leite (eds.), Seminário Lógica no Avião, Brasilia: Universidade de Brasilia: 135-148.
- Faris, John Acheson (1996) The Paradoxes of Zeno (Avebury Series in Philosophy).
- Fearn, Nicholas (2001) Zeno and the Tortoise: How to Think Like a Philosopher. London, Atlantic Books.
- Hager, Paul (1987) Russell and Zeno's Arrow Paradox. Russell: The Journal of Bertrand Russell Studies 7 (1):3.
- McCoy, Casey D. (2018) On Classical Motion, Philosophers' Imprint 18.
- Ruzsa, Imre (1966) On some philosophical problems of mathematics. Bp., Tankönyvkiadó (Hungarian).
- Sainsbury, Richard Mark (2012) Paradoxes. trans. Ferenc Csaba, Bp.,Typotext.
- Salmon, Wesley Charles (ed.) (1970) Zeno's Paradoxes. Indianapolis, IN, USA:Bobbs-Merril.