Can we get in heaven with a hat on?

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Abstract

At the gates of heaven, a long queue has formed, with each person assigned a number to prevent disputes. Saint Peter will only open the gate if the following condition is met: *A person wears a hat if and only if no one standing behind them wears a hat.*
I will demonstrate that Saint Peter will never need to open the gate if the queue is infinite–unless a miraculous individual, who does not simply stand at the end after an infinite sequence of waiting souls, declares: “I am ω, the smallest transfinite ordinal number.”

At the entrance to heaven, there is a long queue; everyone has a number to avoid quarrels. Saint Peter will open the gate only if the following rule is fulfilled:

(i) *Anyone has a hat on his head if and only if no one behind him has a hat on his head.*

I will prove that Saint Peter need not bother to open the gate if there is an infinite queue at the gate, and someone—who is capable of such miracles—does not stand there last after an infinite number of people waiting, saying: "I am ω, the smallest transfinite ordinal number." (Zermelo-Fraenkel set theory miraculously has such a number.)

Consider the case of a single person at the gate. This is how he might reason: If there were a hatted person behind me, I would not put my hat on; but because there is no one standing behind me, there is no hatted person. Therefore, the opposite case applies, so I must put my hat on.

Now consider the case of two people in line. The last person in line would still apply the same reasoning and conclude that he should wear a hat. Consequently, the person in front of him would be left without a hat. If there are three people, the first two would not have hats on their heads, while only the last one would wear a hat. This can be represented as follows:

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Is the following configuration possible?

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No, because the second person could only have a hat on his head if no one behind him had a hat. But since the person behind him does, he cannot wear one either. Is the following configuration possible?

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Let us apply the rule. The first person does not wear a hat because the person behind him does. The second person wears a hat, and the person behind him does not. The rule is fulfilled because there is no hatted person behind the second one. However, the third person has no hat, breaking the rule because there is no hatted person behind him, so the third person must put on a hat. The moment the third one puts on a hat, the second one must remove his hat because there is now a hatted person behind him. Therefore, this configuration is not possible either. Let the number '1' denote a hatted person and '0' a hatless one. It can then be seen that only the sequence $\langle 000 \ldots 0001 \rangle$ satisfies the rule—that is, only the last person can have a hat on. The question, however, is whether there is a last member of the sequence to base an evaluation on.

What is the general mathematical formulation of this problem? The question is whether there exists a function f that satisfies the following condition:

(1) $\forall n \forall k : n, k$ is a natural number $\rightarrow (1 = f(n) \leftrightarrow (n < k \rightarrow 0 = f(k)))$

I can prove that no such function exists when the sequence is restricted to natural numbers. Suppose, on the contrary, that such a function f_o exists. Then n is either hatted or not. For any given n, let hatted $= f_o(n)$. If n is hatted, then everyone behind it is hatless; however, $n + 1$, immediately behind it, could wear a hat. If $n + 1$ puts on a hat, the one in front of him cannot have a hat. Thus, n must be hatless: no hatted $= f_o(n)$. This leads to a contradiction; therefore, the assumption that f_o exists must be rejected. Thus, n must be hatless. However, this is true for any n; therefore, no one can have a hat. Then, paradoxically, anyone can wear a hat, leading to another contradiction. Because f_o was arbitrary, it follows that such a function does not exist.

This result aligns with the principles illustrated by the Yablo paradox. The situation changes if the domain of interpretation contains the smallest transfinite ordinal number ω in addition to the natural numbers. In this case, the last member of the sequence, ω, is the first transfinite ordinal number greater than any natural number.

In this context, there exists a function f_2 such that for every natural number, the function has a value of 0 (= hatless); whereas for the first transfinite ordinal, the function has a value of 1 (= hatted); see Figure 1.

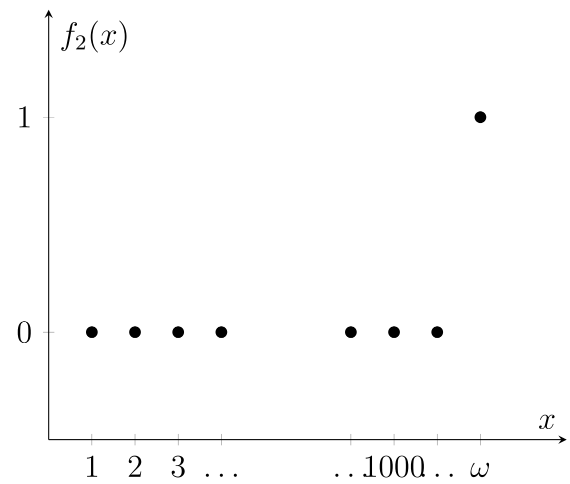

Figure 1: f_2 function

Swapping the variables k and n in the '<' relation in Formula (1) yields the following formula:

(2) $\forall n \forall k : n, k$ is a natural number $\rightarrow (1 = f(n) \leftrightarrow (k < n \rightarrow 0 = f(k)))$

In natural language, this means the following rule:

(ii) *Anyone has a hat on his head if and only if no one before him has a hat on his head.*

The reasoning for this rule can be outlined as follows: The first person in line observes that no one is ahead of him, so there is no hatted person before him. Consequently, he must put his hat on. The second person sees that the individual in front of him wears a hat, so he cannot wear a hat. The third person sees no hat on the head of the person immediately in front of him, and because everyone follows the rule, it follows that someone farther ahead must have a hat. Therefore, he cannot wear a hat either. This reasoning extends to everyone in line. It can then be seen that only the sequence $\langle 100\ldots000\rangle$ satisfies the (ii) rule—that is, only the first person can have a hat on. In the previous case, the existence of the series depends on whether there is a last member to base an evaluation on. Here, it depends on whether there is a first member to base the reasoning on. Figure 2 shows the function defined by Formula (2).

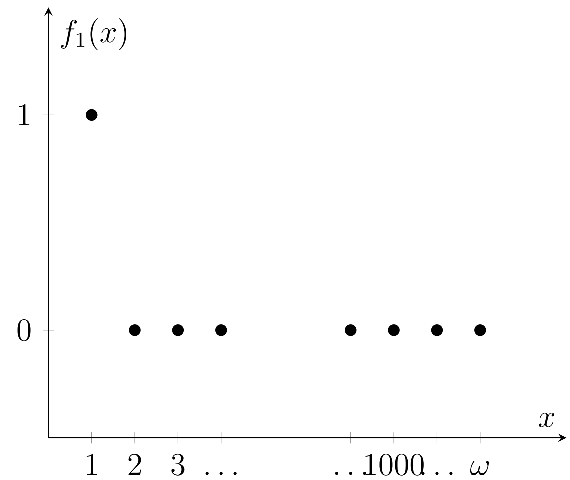

Figure 2: f_1 function

Restricting the domain of interpretation to natural numbers, Formula (2) defines an infinite series, whereas Formula (1) does not. The essential question is: Why does the second rule define a series while the first rule does not?

The “hatted” example was inspired by the papers of Graham Priest and Roy A. Sorensen.