From Zeno to Einstein

Ferenc András

ORCID iD: 0009-0002-9995-8173

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Abstract

Some experimental theories of quantum gravity, such as loop quantum gravity, propose a discrete or "quantized" structure for space-time at very small scales. These theories suggest that space-time is fundamentally composed of discrete units or "atoms" of space, analogous to how matter is made up of elementary particles. In Einstein's special theory of relativity, there exists a maximum speed limit, beyond which no massive particle in an inertial reference frame can travel. In this paper, I will argue that, in addition to the existence of this upper velocity limit, there are logical reasons to consider space and time as having a discrete, atomic structure. This argument draws upon the fundamental insights provided by Zeno's famous aporias.

Keywords: philosophy of physics, special theory of relativity, space-time, discrete space-time, Zeno's aporias

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Introduction

In Einstein's special theory of relativity, it is asserted that there is a maximum speed limit, beyond which no point of mass in an inertial reference frame can travel. In this paper, I will argue that, in an inertial reference frame, there are also logical reasons to consider space and time as having an atomic structure. The basic idea of this argument was suggested by Zeno's well-known aporias.[1]

[1]The first version of this argument appeared (in Hungarian) as part of a discussion paper in Hungarian Physical Review (András, 2005.). My source for Zeno's aporias is Ruzsa, 1966.

Premises

Suppose that a mass point m travels along a straight line from point A to point B, starting at time t_1 and arriving at time t_2.

(1) We assume that both the distance travelled and the corresponding time are discrete rather than continuum. This implies that there is an isomorphic relationship between the domains of space and time atoms and the integers ordered by magnitude. In a given inertial frame:

- For any two instants t_1 and t_2, either t_1 precedes t_2, t_2 precedes t_1, or t_1 equals t_2.
- This allows space and time coordinates to be represented as sequences of integers.

From this assumption, the following sub-premises emerge:

(1.1) There are finitely many points between A and B, with each having a clearly defined predecessor and successor. For any point $x, x+1$ is the next point, and $x-1$ is the preceding one.

(1.2) The points along the route of m are ordered such that, for any two distinct points, we can determine which is closer to B.

(2) A point exists at all times, meaning it occupies a specific location at every moment.

(3) If the point is somewhere, it exists at a specific time.

(4) The point m moves from one point to the next (or previous) point. The set of potential next locations at any given time is referred to as "adjacent places". This concept extends naturally to higher-dimensional discrete spaces, as discussed later.

(5) In the context of forward motion, the point never moves backward in time. Thus, m touches all places between A and B, skipping none.

In deterministic systems only:

(6) At any given time, the point is at one and only one location. This location is a deterministic property of the point. Alternatively, in a probabilistic framework, the location of m might be interpreted as a function of a statistical distribution.

Let us further assume:

(7) A point is continuously advancing, so that if x is later than y, then the position of the point at x is closer to B than the position at the earlier time y. Thus, it follows that the point moves continuously through the range between t_1 and t_2, including t_1 and t_2, as it travels. In continuous motion, if the two times are not the same, the two corresponding points of the point are also different.

Conclusion

The point can only travel at a single speed—the maximum speed determined by the shortest possible distance and time interval.

Proof

Suppose that, within one time atom, a point travels across two or more spatial atoms. Given that the point must occupy a position at all times and cannot exist in multiple places simultaneously, this would result in ambiguity regarding its location on a discrete time scale. This contradicts the assumption that the point moves through space in a sequential and well-defined manner.

Conversely, if the point travels slower than the maximum speed, there will be instances where it occupies the same spatial location at consecutive time steps. This violates the premise of continuous progress.

However, if continuous progress is not required, slower velocities may correspond to intermittent motion. In such cases, the point may remain stationary for certain intervals before advancing. This allows for velocities lower than the maximum speed but introduces a non-continuous pattern of movement. (See figure 1 below.)

Proof in formal logic language

Notations:

$S(x) :=$ x is a location

$S(x) \wedge S(y) \wedge x < y :=$ y is further from the origo than x

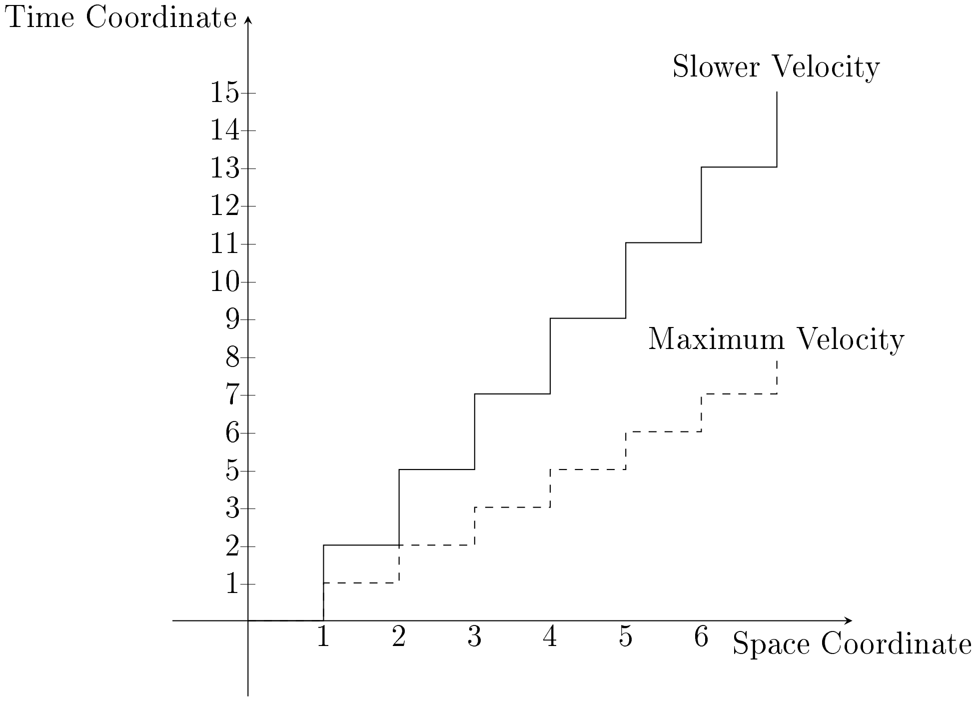

Figure 1: World lines in atomic space-time

$T(x) :=$ x is a time atom

$T(x) \wedge T(y) \wedge x < y :=$ y later than x

s_m is the time-distance function of a mass point m

$y = s_m(x) :=$ distance of m mass point is y at x time

Premises:

(1) $\forall x \forall y \forall z(\neg x < x \wedge (x < y \rightarrow \neg y < x) \wedge ((x < y \wedge y < z) \rightarrow x < z))$

The relation $<$ is both asymmetric and transitive, making it a strict ordering.

(2) $S(A) \wedge S(B) \wedge \neg \exists x(S(x) \wedge x < A) \wedge \neg \exists x(S(x) \wedge B < x)$

A and B are boundary points in a defined spatial domain.

(3) $\forall x \forall y((T(x) \wedge T(y)) \rightarrow (x < y \vee y < x \vee x = y))$

(4) $\forall x \forall y((S(x) \wedge S(y)) \rightarrow (x < y \vee y < x \vee x = y))$

Any two time or place atoms are comparable.

(5) $\forall x(x-1 < x \wedge x < x+1) \wedge \forall x \neg \exists y(x < y \wedge y < x+1) \wedge \forall x \neg \exists y(x-1 < y \wedge y < x) \wedge \forall x(x \neq x-1) \wedge \forall x(x \neq x+1))$

Space and time have an atomic structure.

(6) $\forall y(S(y) \rightarrow \exists x(y = s_m(x) \wedge T(x))) \wedge \forall x(T(x) \rightarrow \exists y(y = s_m(x) \wedge S(y)))$

(7) $\forall x \forall y \forall z((S(x) \wedge S(y) \wedge S(z) \wedge x < y \wedge y < z) \rightarrow$

$\exists u \exists v \exists w(T(u) \wedge T(v) \wedge T(w) \wedge x = s_m(u) \wedge y = s_m(v) \wedge z = s_m(w) \wedge u < v \wedge v < w))$

The moving point does not miss or skip places.

(8) $\forall x \forall y((T(x) \wedge T(y) \wedge x < y) \rightarrow s_m(x) < s_m(y))$

A point is continuously advancing.

Deductions:

(I)	Suppose that m moves more than one distance atom in a period:	
(9)	$y = s_m(x) \wedge z = s_m(x+1) \wedge y+1 < z$	
	This is impossible because it contradicts (5)(7).	
(9.1)	$a = s_m(x) \wedge b = s_m(x+1) \wedge a+1 < b$	(9)
(9.2)	$a < a+1 \wedge a+1 < b$	(5)(9.1)
(9.3)	$a = s_m(x) \wedge b = s_m(x+1)$	(9.1)
(9.4)	$\exists v(a = s_m(x) \wedge a+1 = s_m(v) \wedge b = s_m(x+1))$	(7)(9.3)(9.2)
(9.5)	$\exists v(x < v \wedge v < x+1)$	(7) (9.4)
(9.6)	$\neg \exists v(x < v \wedge v < x+1)$	(5)
(9.7)	$\neg \exists v(x < v \wedge v < x+1) \wedge \exists v(x < v \wedge v < x+1)$	(9.5) (9.6)

(II)	Suppose that m passes less than one atom distance.	
(10)	$y = s_m(x) \wedge z = s_m(x+1) \wedge y < z \wedge z < y+1$	
	This is impossible because it contradicts (5).	
(10.1)	$a = s_m(x) \wedge b = s_m(x+1) \wedge a < b \wedge b < a+1$	(10)
(10.2)	$\neg \exists y(a < y \wedge y < a+1)$	(5)
(10.3)	$\exists y(a < y \wedge y < a+1)$	(10.1)
(10.4)	$\neg \exists y(a < y \wedge y < a+1) \wedge \exists y(a < y \wedge y < a+1)$	(10.2)(10.3)

(III) Suppose that m does not move in a range of time.

(11) $y = s_m(x) \wedge y = s_m(x+1)$

This is impossible because it contradicts (8).

(11.1)	$a = s_m(1) \wedge a = s_m(2)$	(11)
(11.2)	$s_m(1) < s_m(2)$	(8)(11.1)
(11.3)	$a < a$	(11.1)(11.2)
(11.4)	$\neg a < a$	(1)
(11.5)	$\neg a < a \wedge a < a$	(11.3)(11.4)

(IV) The remaining possibility is that m mass point move continuously with maximum velocity.

(12) $\forall x \forall y((T(x) \wedge S(y)) \rightarrow (y = s_m(x) \rightarrow y + 1 = s_m(x+1)))$

As I noted earlier, if continuous progress is discarded, slower velocities can be matched by the intermittent advancement of the point and interspersed with times where it remains in place for longer or shorter periods of time, depending on the velocity. Thus, the point may travel more slowly than the maximum speed in such a special intermittent manner.

Open questions

In the figure 1, I use a regular straight line to plot coordinates. However, if distance truly consists of discrete, atomistic points, the regular arrangement of these spatial atoms is not evident. Only condition (1)–(7) governs the logic of discrete spatial atoms. From the perspective of a continuous world, the spatial atoms of a discrete world might not align along a straight line but could instead form a wave-like arrangement. Consequently, a square grid representing one-dimensional space-time does not necessarily conform to a regular geometric shape when viewed from a continuous framework.

Imagine drawing a square grid of one-dimensional space-time on a rubber sheet that can be stretched or compressed. The logical requirements are still satisfied, as long as grid points in space-time are not skipped. This is why I depict maximum speed using a stair-step graph rather than a straight line.[2] The significance of this becomes clearer when extended to two-dimensional space.

[2]The consideration of grid points is inspired by a 2006 lecture by Tamás Sándor Bíró at the Skeptic Conference.

Why do we assume that the distance between two points is represented by a straight line and not a curved one? This is because distance is conventionally defined as the shortest path between two points, with a straight line being the shortest possible route. For simplicity, let us consider regular layouts. Assume the twelve atomic locations in a two-dimensional discrete world are arranged as follows (See table 1 below.):

A	E	I
B	F	J
C	G	K
D	H	L

Table 1: Two dimensional finite space

How far are points A and K? The answer depends on the rules for movement. In point (4), I stated that in one-dimensional space, motion can only occur to adjacent locations at consecutive, discrete times. While adjacency is straightforward in one dimension, in two- or multi-dimensional space, there are multiple paths to choose from.

If diagonal movement is allowed, touching F, the shortest distance between A and K involves two atomic steps. If only horizontal and vertical moves are allowed, the distance becomes four atomic steps.

What about the distance between A and D? Intuitively, we might say they are three atomic distances apart. But is this truly the shortest path? Again, the answer depends on the movement rules. If diagonal movement is allowed, we can traverse $A \to E \to D$, reducing the distance to two atomic steps. However, if only horizontal and vertical moves are permitted, the shortest path requires three atomic steps. This raises another question: Is diagonal movement, such as $A \to F$, permissible? A similar inquiry applies to three-dimensional space. How far apart are A and L? If diagonal movement is allowed but jumping is prohibited, there are three distinct routes of three atomic steps:

- $A \to B \to G \to L$
- $A \to F \to K \to L$
- $A \to F \to G \to L$

If diagonal movement is disallowed, the shortest path between A and L becomes five atomic steps. In either case, the Pythagorean theorem does not hold, nor does subdividing distances into many smaller points

resolve the issue. (See McDaniel, 2007.)
A continuous space-time model could help resolve these dilemmas. Perhaps such a model could be mapped onto a probabilistic atomic space-time framework, where random information replaces the excess precision of continuous quantities (real numbers). This could offer a potential solution to the challenges posed by discrete atomic space-time.

Epilogue

Currently, there are no widely accepted physical theories that definitively posit an atomic structure for space-time. In classical physics, the dominant perspective holds that space and time are continuous and infinitely divisible. This concept is deeply embedded in Einstein's general relativity, where gravity is described as the curvature of space-time.

However, some experimental theories in the domain of quantum gravity, such as loop quantum gravity, challenge this continuous framework by suggesting that space-time may have a fundamentally discrete structure at extremely small scales. These theories propose that space-time consists of discrete units or "atoms" of space, analogous to how matter is composed of particles.

One of the key predictions of these theories is the existence of a minimum time interval, known as the Planck time, which is approximately 10^{-43} seconds. The corresponding Planck distance—the distance light travels in one unit of Planck time—is estimated to be around 10^{-35} meters. Such scales are far beyond current experimental capabilities, but they offer a fascinating glimpse into the potential granularity of space-time, providing a bridge between the quantum and relativistic realms.

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