# The Definite Story on Yablo's Paradox

## Why all subsequent papers on this matter are vain


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PALMYR 2007

# Yablo's paradox

- Let $\varphi_0, \varphi_1, \ldots, \varphi_n, \ldots$ be propositions such that
  $$\varphi_n \text{ iff, } \forall m > n,\ \varphi_m \text{ is false}$$
- Leads to a contradiction!
  - If $\varphi_0$ is true then $\varphi_1$ and $\varphi_2, \varphi_3, \ldots$ are false. But if $\varphi_2, \varphi_3, \ldots$ are false, then $\varphi_1$ is true
  - If $\varphi_0$ is false, then there exists n such that $\varphi_n$ is true. But then, by same argument, $\varphi_{n+1}$ is false and true.
- A new kind of paradox:
  - Yablo (1993, 2004): No self-reference
  - Tennant (1995): Proof features not circular, regressive
  - Ketland (2005): omega-paradox

# Questions

- General questions
  - Is YP truly a truth-theoretical paradox? Can we prove the existence of the sequence of sentences?
  - If so, is YP non circular? How do we characterise this?
- Specific question
  - What can be said of descending hierarchies of truth predicates (DHTPs) with YP?

# Priest's analysis of YP

- Prefers the finitary formulation using a satisfaction predicate Sat:

  Sat($n$,$s$) iff $n$ satisfies unary predicate $s$

  i.e. $s$ is the code number of a unary predicate.
- A Yablo sequence is a predicate $Y(x)$ such that

  (Y) $\forall x(Y(x) \leftrightarrow \forall y > x\, \neg \mathrm{Sat}(y, \ulcorner Y(x) \urcorner)$
- Remarks:
  - Finitary contradiction
  - $Y(x)$ fixed point

# Priest continued

- Answer to first general question:
  - Existence of $Y(x)$ can be proven using general diagonal lemma. YP is truth-theoretical!
- Answer to second:
  - Must define what a self-referential paradox is. For Priest, inclosure schema (Russell 1903; Priest 1995, 1997).
  - YP can be made into an inclosure schema. Therefore, YP is non novel truth-theoretical paradox.

# Remarks on Priest

- Ketland (2005): Can be made into omega-paradox if we take numerical instances of $Z(x)$, where
  $$Z(x) \leftrightarrow \forall y > x \neg T(\ulcorner Z(y) \urcorner)$$
- Satisfaction predicate is liar-like:
  - The fixed point $Y(x)$ of (Y) contains Sat
  - Priest's YP is a liar-like paradox
- Can we still make YP into a paradox if we require that it respects some hierarchy?
- If so, is this a novel paradox?

# Descending Hierarchies of Truth Predicates

- Could the novel character of YP manifest itself here? (Forster 1996; Yablo 2004)
- Reframing Yablo for DHTPs:
  - $L$ language of $PA$
  - $L^{\mathrm{T}}$ language $L + T_0, T_1, T_2, \ldots$
  - $PAT = PA + \{T_n(\ulcorner\varphi\urcorner) \leftrightarrow \varphi : \varphi \in \mathbf{Sent}_{n+1}\}$
- A Yablo sequence will be $\varphi_0, \varphi_1, \varphi_2, \ldots$ such that, for all $n$,
  - $\varphi_n \in \mathbf{Sent}_n$
  - $PAT \vdash \varphi_n \leftrightarrow \forall x > \underline{n}\ \neg T_n(\ulcorner\varphi_x\urcorner)$

# Yablo as non-wellfoundedness

- Can perhaps show with YP that DHTPs are (omega)-inconsistent?
  **Proposition.** If there exists a Yablo sequence, then *PAT* is omega inconsistent.
- We already know that
  **Theorem (Visser).** *PAT* is omega inconsistent.
- Could we strengthen this to
  **Theorem.** There exists a Yablo sequence.
- We will see, but for now…

# What about YP, set theory and well-foundedness?

- There is a conceptual isomorphism between truth-theoretical paradoxes and set-theoretical ones
- What corresponds to YP in set theory?
- Goldstein (2004): $\{G_n : n \in \mathbb{N}\}$ such that

$$x \in G_n \text{ iff, for all } m > n, x \notin G_m$$

- Fixed point of the function $\mathcal{Y}$

$$\{X_n : n \in \mathbb{N}\} \mapsto \{\mathcal{Y}(X_n) : n \in \mathbb{N}\}$$

where

$$x \in \mathcal{Y}(X_n) \text{ iff, for all } m > n, x \notin X_m$$

- Existence problem is still there.

# Yablo as Mirimanoff

- Exploiting another strategy: similarity between liar and Russell
- $n$-Russell and $n$-liar:
  - $\{x : \forall y_1,\ldots,y_n \neg(x \in y_1 \in y_2 \in \ldots \in y_n \in x)\}$
  - For all $k = 0,1,\ldots,n-1$,

    $\varphi_k$ iff $\varphi_l$ is false, $l > k$

    $\varphi_n$ iff $\varphi_l$ is true, $l < n$
- When $n \to \infty$:
  - ω-Russell = Mirimanoff (∈-wellfounded sets)
  - ω-liar = Yablo

# Mirimanoff and TNT

- Non-wellfoundedness in a typed hierarchy of sets would be analogue to YP
- Yablo (2004) sketches inconsistency in typed set theory with negative types (TNT), working on Wang (1953)
- In TNT, can form some Mirimanoff paradox:
  - $W_n$ = sets of type $n$ that are $\in$-well-founded
  - $W_n$ is of type $n$+1
  - If $W_n \in W_{n+1}$ then … $W_{n\text{-}1} \in W_n \in W_{n+1}$
  - If $W_n \notin W_{n+1}$ then an element of $W_n$ is $\in$-non-well-founded
- This is not formalisable in TNT. Should have been clear from the start since TST consistent iff TNT is.

# Mirimanoff and TNT$^+$

- Let TNT$^+$ be the infinitary version of TNT, i.e.
  - We take the language of TNT$^+$ to be the infinitary language $L_{\omega 1 \omega 1}(\in,=)$
  - Add suitable infinitary logical axioms and rules
  - Still not enough. Must add infinitary comprehension (IC).
- TNT$^+$ is inconsistent (in this infinitary logic).

$$\forall y_1,\ldots,y_n,\ldots \neg(\bigwedge_n (y_{n+1} \in y_n) \wedge y_1 \in x)$$

- The argument does not apply to TST$^+$ because of well founded-ness of TST.
- Does not apply to ZFC$^+$ but this has nothing to do with $\in$-well-foundedness however.
- Could it apply to infinitary new foundations?

# Wellfoundedness and $NF^{+}$

- Is NF set theory with infinitary touch in the same predicament?
- NF like naïve set theory except for stratification requirement for formulas in comprehension:
  - Must be possible to number the variables so that resulting atomic expressions are of the form $n \in n+1$ and $m = m$
- Define infinitary stratification as the existence of a numbering (with elements of the cardinal number of **Var**) with same properties as above.
- Then formula

  $$\forall y_1,\ldots,y_n,\ldots \neg(\bigwedge_n (y_{n+1} \in y_n) \wedge y_1 \in x)$$

  is not stratified. Non-$\in$-well-founded formulas not stratified. Comprehension cannot be applied to them.

# Visser's result

- How does non-well-foundedness fare with DHTPs?
- Visser (1989): Not well. *PAT* is ω-inconsistent.
- First step:
  - Define recursive function *f* such that changes Gödel numbers of formulas like
    $$f(\ulcorner T_n(t)\urcorner) = \ulcorner T_{n+1}(f^*(t))\urcorner$$
    and modifies rest accordingly.
  - $f^*$ represents *f* in *PA*
- Second step:
  - Define $\varphi^{(m)}$ as the formula to which *f* has been applied *m* times ($\varphi$'s truth predicate has been pushed down *m* levels).
  - Take χ to be fixed point $\forall x > 0\ \neg T_0(\ulcorner y^{(x)}\urcorner)$, i.e.

$$\chi \leftrightarrow \forall x > 0\ \neg T_0(\ulcorner \chi^{(x)}\urcorner)$$

# Visser's $\chi^{(n)}$ is a Yablo sequence!

- The Yablo sequence is $\chi^{(n)}$, we have

$$\begin{aligned}\chi^{(n)} &\leftrightarrow (\forall x > 0\ \neg T_0(\ulcorner \chi^{(x)} \urcorner))^{(n)} \\ &\leftrightarrow \forall x > 0\ \neg T_n(\ulcorner \chi^{(x+n)} \urcorner) \\ &\leftrightarrow \forall x > \underline{n}\ \neg T_n(\ulcorner \chi^{(x)} \urcorner)\end{aligned}$$

  and each $\chi^{(n)} \in \mathbf{Sent}_n$
- This Yablo sequence is finitarily stated but leads only to omega-inconsistency (not inconsistency).

# **Postscript:** Explicit construction of a non standard model of *PAT*

- For each $N$, one can construct a standard model $M^N$ of a finitely descending truth theory on $PA$ with truth predicates $T_0, T_1, \ldots, T_N$
- $M^N$ satisfies

$$\alpha^N = \{T_n(\ulcorner \varphi \urcorner) \leftrightarrow \varphi : \varphi \in \mathbf{Sent}_{n+1,N}\}_{0 \leq n \leq N}$$

where $\mathbf{Sent}_{k,N}$ is $\mathbf{Sent}_k$ truncated at $N$.

- Add empty extensions for predicates $T_k$ for $k > N$ to $M^N$ to get an $L^{\mathrm{T}}$-structure $M_N$ satisfying $\alpha^N$.

# Postscript continued

- Choose a non-principal ultrafilter $U$ over $\mathbb{N}$
- Ultraproduct the models $M_n$ to get $\mathcal{M} = \prod_U M_n$.
- $\mathcal{M}$ is an extension of the ultrapower $\mathcal{N}^U$ of the standard model $\mathcal{N}$.
- $\mathcal{M}$ satisfies $PAT$ because $\mathcal{N}^U$ satisfies $PA$ and $\mathcal{M}$ satisfies all the $\alpha^n$.

# Open questions

- Thought we could take the diagonal submodel of $\delta(\mathcal{M})$ to define a standard model of *PAT*?
- **Wrong:** nothing guarantees $\delta(\mathcal{M})$ satisfies the *T*-schemas of *PAT*.
- **Question 1:** Just how many *T*-schemas of *PAT* does $\delta(\mathcal{M})$ satisfy?
- **Question 2:** Is there some "well-foundedness" condition we can define on formulas such that, for these formulas, the T-schemas are valid in $\delta(\mathcal{M})$?