

# The Definite Story on Yablo's Paradox

Why all subsequent papers on this matter are vain



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# Yablo's paradox

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- Let  $\varphi_0, \varphi_1, \dots, \varphi_n, \dots$  be propositions such that  
 $\varphi_n$  iff,  $\forall m > n, \varphi_m$  is false
- Leads to a contradiction!
  - If  $\varphi_0$  is true then  $\varphi_1$  and  $\varphi_2, \varphi_3, \dots$  are false. But if  $\varphi_2, \varphi_3, \dots$  are false, then  $\varphi_1$  is true
  - If  $\varphi_0$  is false, then there exists  $n$  such that  $\varphi_n$  is true. But then, by same argument,  $\varphi_{n+1}$  is false and true.
- A new kind of paradox:
  - Yablo (1993, 2004): No self-reference
  - Tennant (1995): Proof features not circular, regressive
  - Ketland (2005): omega-paradox

# Questions

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- General questions

- Is YP truly a truth-theoretical paradox? Can we prove the existence of the sequence of sentences?
- If so, is YP non circular? How do we characterise this?

- Specific question

- What can be said of descending hierarchies of truth predicates (DHTPs) with YP?

# Priest's analysis of YP

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- Prefers the finitary formulation using a satisfaction predicate Sat:

Sat( $n, s$ ) iff  $n$  satisfies unary predicate  $s$   
i.e.  $s$  is the code number of a unary predicate.

- A Yablo sequence is a predicate  $Y(x)$  such that

$$(Y) \quad \forall x(Y(x) \leftrightarrow \forall y > x \neg \text{Sat}(y, \ulcorner Y(x) \urcorner))$$

- Remarks:
  - Finitary contradiction
  - $Y(x)$  fixed point

# Priest continued

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- Answer to first general question:
  - Existence of  $Y(x)$  can be proven using general diagonal lemma. YP is truth-theoretical!
- Answer to second:
  - Must define what a self-referential paradox is. For Priest, inclosure schema (Russell 1903; Priest 1995, 1997).
  - YP can be made into an inclosure schema. Therefore, YP is non novel truth-theoretical paradox.

# Remarks on Priest

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- Ketland (2005): Can be made into omega-paradox if we take numerical instances of  $Z(x)$ , where

$$Z(x) \leftrightarrow \forall y > x \neg T(\ulcorner Z(y) \urcorner)$$

- Satisfaction predicate is liar-like:
  - The fixed point  $Y(x)$  of  $(Y)$  contains Sat
  - Priest's YP is a liar-like paradox
- Can we still make YP into a paradox if we require that it respects some hierarchy?
- If so, is this a novel paradox?

# Descending Hierarchies of Truth Predicates

- Could the novel character of YP manifest itself here? (Forster 1996; Yablo 2004)
- Reframing Yablo for DHTPs:
  - $L$  language of  $PA$
  - $L^T$  language  $L + T_0, T_1, T_2, \dots$
  - $PAT = PA + \{T_n(\ulcorner \varphi \urcorner) \leftrightarrow \varphi : \varphi \in \mathbf{Sent}_{n+1}\}$
- A Yablo sequence will be  $\varphi_0, \varphi_1, \varphi_2, \dots$  such that, for all  $n$ ,
  - $\varphi_n \in \mathbf{Sent}_n$
  - $PAT \vdash \varphi_n \leftrightarrow \forall x > \underline{n} \neg T_n(\ulcorner \varphi_x \urcorner)$

# Yablo as non-wellfoundedness

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- Can perhaps show with YP that DHTPs are (omega)-inconsistent?

**Proposition.** If there exists a Yablo sequence, then *PAT* is omega inconsistent.

- We already know that

**Theorem (Visser).** *PAT* is omega inconsistent.

- Could we strengthen this to

**Theorem.** There exists a Yablo sequence.

- We will see, but for now...



# What about YP, set theory and well-foundedness?

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- There is a conceptual isomorphism between truth-theoretical paradoxes and set-theoretical ones
- What corresponds to YP in set theory?
- Goldstein (2004):  $\{G_n : n \in \mathbb{N}\}$  such that
$$x \in G_n \text{ iff, for all } m > n, x \notin G_m$$
- Fixed point of the function  $\mathcal{Y}$ 
$$\{X_n : n \in \mathbb{N}\} \mapsto \{\mathcal{Y}(X_n) : n \in \mathbb{N}\}$$
where
$$x \in \mathcal{Y}(X_n) \text{ iff, for all } m > n, x \notin X_m$$
- Existence problem is still there.

# Yablo as Mirimanoff

- Exploiting another strategy: similarity between liar and Russell
- $n$ -Russell and  $n$ -liar:
  - $\{x : \forall y_1, \dots, y_n \neg (x \in y_1 \in y_2 \in \dots \in y_n \in x)\}$
  - For all  $k = 0, 1, \dots, n - 1$ ,
    - $\varphi_k$  iff  $\varphi_l$  is false,  $l > k$
    - $\varphi_n$  iff  $\varphi_l$  is true,  $l < n$
- When  $n \rightarrow \infty$ :
  - $\omega$ -Russell = Mirimanoff ( $\in$ -wellfounded sets)
  - $\omega$ -liar = Yablo

# Mirimanoff and TNT

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- Non-wellfoundedness in a typed hierarchy of sets would be analogue to YP
- Yablo (2004) sketches inconsistency in typed set theory with negative types (TNT), working on Wang (1953)
- In TNT, can form some Mirimanoff paradox:
  - $W_n$  = sets of type  $n$  that are  $\in$ -well-founded
  - $W_n$  is of type  $n+1$
  - If  $W_n \in W_{n+1}$  then...  $W_{n-1} \in W_n \in W_{n+1}$
  - If  $W_n \notin W_{n+1}$  then an element of  $W_n$  is  $\in$ -non-well-founded
- This is not formalisable in TNT. Should have been clear from the start since TST consistent iff TNT is.

# Mirimanoff and TNT<sup>+</sup>

- Let TNT<sup>+</sup> be the infinitary version of TNT, i.e.
  - We take the language of TNT<sup>+</sup> to be the infinitary language  $L_{\omega_1\omega_1}(\in, =)$
  - Add suitable infinitary logical axioms and rules
  - Still not enough. Must add infinitary comprehension (IC).
- TNT<sup>+</sup> is inconsistent (in this infinitary logic).
$$\forall y_1, \dots, y_n, \dots \neg (\bigwedge_n (y_{n+1} \in y_n) \wedge y_1 \in x)$$
- The argument does not apply to TST<sup>+</sup> because of well founded-ness of TST.
- Does not apply to ZFC<sup>+</sup> but this has nothing to do with  $\in$ -well-foundedness however.
- Could it apply to infinitary new foundations?

# Wellfoundedness and $NF^+$

- Is NF set theory with infinitary touch in the same predicament?
- NF like naïve set theory except for stratification requirement for formulas in comprehension:
  - Must be possible to number the variables so that resulting atomic expressions are of the form  $n \in n+1$  and  $m = m$
- Define infinitary stratification as the existence of a numbering (with elements of the cardinal number of **Var**) with same properties as above.
- Then formula
$$\forall y_1, \dots, y_n, \dots \neg (\bigwedge_n (y_{n+1} \in y_n) \wedge y_1 \in x)$$
is not stratified. Non- $\in$ -well-founded formulas not stratified. Comprehension cannot be applied to them.

# Visser's result

- How does non-well-foundedness fare with DHTPs?
- Visser (1989): Not well. *PAT* is  $\omega$ -inconsistent.
- First step:
  - Define recursive function  $f$  such that changes Gödel numbers of formulas like
$$f(\ulcorner T_n(t) \urcorner) = \ulcorner T_{n+1}(f^*(t)) \urcorner$$
and modifies rest accordingly.
  - $f^*$  represents  $f$  in *PA*
- Second step:
  - Define  $\varphi^{(m)}$  as the formula to which  $f$  has been applied  $m$  times ( $\varphi$ 's truth predicate has been pushed down  $m$  levels).
  - Take  $\chi$  to be fixed point  $\forall x > 0 \neg T_0(\ulcorner \varphi^{(x)} \urcorner)$ , i.e.

$$\chi \leftrightarrow \forall x > 0 \neg T_0(\ulcorner \chi^{(x)} \urcorner)$$

# Visser's $\chi^{(n)}$ is a Yablo sequence!

- The Yablo sequence is  $\chi^{(n)}$ , we have

$$\begin{aligned}\chi^{(n)} &\leftrightarrow (\forall x > 0 \neg T_0(\ulcorner \chi^{(x)} \urcorner))^{(n)} \\ &\leftrightarrow \forall x > 0 \neg T_n(\ulcorner \chi^{(x+n)} \urcorner) \\ &\leftrightarrow \forall x > \underline{n} \neg T_n(\ulcorner \chi^{(x)} \urcorner)\end{aligned}$$

and each  $\chi^{(n)} \in \mathbf{Sent}_n$

- This Yablo sequence is finitarily stated but leads only to omega-inconsistency (not inconsistency).

# Postscript: Explicit construction of a non standard model of $PAT$

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- For each  $N$ , one can construct a standard model  $M^N$  of a finitely descending truth theory on  $PA$  with truth predicates  $T_0, T_1, \dots, T_N$
- $M^N$  satisfies
$$\alpha^N = \{T_n(\ulcorner \varphi \urcorner) \leftrightarrow \varphi : \varphi \in \mathbf{Sent}_{n+1, N}\}_{0 \leq n \leq N}$$
where  $\mathbf{Sent}_{k, N}$  is  $\mathbf{Sent}_k$  truncated at  $N$ .
- Add empty extensions for predicates  $T_k$  for  $k > N$  to  $M^N$  to get an  $L^T$ -structure  $M_N$  satisfying  $\alpha^N$ .



# Postscript continued

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- Choose a non-principal ultrafilter  $U$  over  $\mathbb{N}$
- Ultraproduct the models  $M_n$  to get  $\mathcal{M} = \prod_U M_n$ .
- $\mathcal{M}$  is an extension of the ultrapower  $\mathcal{N}^U$  of the standard model  $\mathcal{N}$ .
- $\mathcal{M}$  satisfies *PAT* because  $\mathcal{N}^U$  satisfies *PA* and  $\mathcal{M}$  satisfies all the  $\alpha^n$ .

# Open questions

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- Thought we could take the diagonal submodel of  $\delta(\mathcal{M})$  to define a standard model of *PAT*?
- **Wrong:** nothing guarantees  $\delta(\mathcal{M})$  satisfies the *T*-schemas of *PAT*.
- **Question 1:** Just how many *T*-schemas of *PAT* does  $\delta(\mathcal{M})$  satisfy?
- **Question 2:** Is there some “well-foundedness” condition we can define on formulas such that, for these formulas, the *T*-schemas are valid in  $\delta(\mathcal{M})$ ?