# *Curry, Yablo and duality*

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## 1. *Introduction*

The Liar paradox is (or involves, depending on one's definition of 'paradox') the directly self-referential Liar statement:

This statement is false.

or (where $T$ is a truth predicate):

$\Lambda: \sim T(<\Lambda>)$[1]

The argument that proceeds from the Liar statement and the relevant instance of the $T$-schema:

$T(<\Lambda>) \leftrightarrow \Lambda$

to a contradiction is familiar. In recent years, a number of variations on the Liar paradox have arisen in the literature on semantic paradox. The two that will concern us here are the Curry paradox,[2] and the Yablo paradox.[3]

The Curry paradox demonstrates that neither negation nor a falsity predicate is required in order to generate semantic paradoxes. Given any statement $\Phi$ whatsoever, we need merely consider the statement:

If this statement is true, then $\Phi$

or:

$\Xi: \quad T(<\Xi>) \rightarrow \Phi$

Here, via familiar reasoning, one can 'prove' $\Phi$ merely through consideration of statement $\Xi$ and the $\Xi$-instance of the $T$-schema.

1 Here '<…>' is any appropriate name-forming device, such as Gödel coding. In addition, here and throughout, we shall assume that negation and the truth predicate commute, so that:

$\sim T(<\Phi>)$

and:

$T <\sim\Phi>)$

are inter-derivable (in other words, we assume that falsity is equivalent to failure to be true).

2 The Curry paradox was first formulated in Curry 1942.

3 The Yablo paradox was first formulated in Yablo 1985, although the more familiar presentation is found in Yablo 1993. Yablo himself humbly calls the construction in question the '$\omega$-Liar'.

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Interestingly, the Liar paradox can be viewed as nothing more than a special case of the Curry paradox. If we define negation in terms of the conditional and a primitive absurdity constant '⊥':[4]

$$\sim\Psi =_{df} \Psi \rightarrow \bot$$

then the Liar paradox is simply the instance of the Curry paradox obtained by substituting '⊥' for Φ.

The Yablo paradox demonstrates that circularity is also not required in order to generate semantic paradox.[5] The paradox proceeds by considering an infinite $\omega$-sequence of statements of the form:

$$\begin{array}{ll} S_1: & (\forall m > 1)(\sim T(<S_m>)) \\ S_2: & (\forall m > 2)(\sim T(<S_m>)) \\ S_3: & (\forall m > 3)(\sim T(<S_m>)) \\ \vdots & \vdots \\ S_i: & (\forall m > i)(\sim T(<S_m>)) \\ \vdots & \vdots \end{array}$$

– that is, the set of all statements of the form:

$$S_n: \quad (\forall m > n)(\sim T(<S_m>))$$

where $n$ is a natural number. Again, the reasoning from the statements in the above list and the corresponding $T$-schema instances to a contradiction is well known, and need not be belaboured here.[6]

As in the case of the Curry paradox, strong connections have been drawn between the Liar paradox and paradoxes that have the same general form as the Yablo paradox – that is, paradoxes that take the form of an infinite list of statements, each of which makes claims about statements occurring 'later' in the list. We shall (following e.g. Beall 1999, Sorensen 1998) call such paradoxes 'Yabloesque'. In particular, 'unwinding' theorems for various formal languages show that any circular semantic paradox consisting of finitely many statements from one of the languages in question can be correlated in a non-ad-hoc manner with an infinitary, non-circular

4 To see why we must use a primitive absurdity constant here, and cannot instead substitute an arbitrary contradiction of the form '$P \wedge \sim P$', the reader should consult Cook and Cogburn 2000.

5 The reader who remains stubbornly convinced that the Yablo paradox is circular in some substantial manner is encouraged to carefully read Cook 2006 and Yablo 2006.

6 Well-known, however, should not be confused with simple. For a guide to the complexities of the reasoning leading to the contradiction, and, in particular, the role of a meta-theoretic version of the $\omega$-rule in the proof, the reader should consult Ketland 2005.

Yabloesque paradox.[7] On these constructions, the Liar paradox is associated with the original Yablo paradox itself.[8]

Thus, there are important connections between the Liar paradox and the Curry paradox, and between the Liar paradox and the Yablo paradox. It would be ideal, then, if we could complete the 'triangle', and find direct connections between the Curry paradox and the Yablo paradox. The obvious first step is to attempt to identify a Curry–Yablo hybrid – that is, replace the self-referential formulation of the Curry paradox given above, which asserts that its own truth entails an arbitrary statement $\Phi$, with an infinite list of statements, each of which asserts that *the statements occurring after it in the list entail* $\Phi$.

There is a complication, however. The difficulty arises when we consider how we ought to interpret the italicized phrase at the end of the previous paragraph. There are (at least) two possibilities here, depending on how we read the scope of the implicit quantifier:

(a) Each statement $S_n$ states that the truth of all $S_m$ occurring after $S_n$ (jointly) entails $\Phi$.
(b) Each statement $S_n$ states that, for any statement $S_m$ occurring after $S_n$, the truth of that statement (individually) entails $\Phi$.

If all we are interested in is generating a Curry-like Yabloesque paradox, then the distinction between (a) and (b) doesn't matter: either reading will (as we shall see below) generate a paradox. And since (at least in non-paraconsistent logics) any two inconsistent theories are equivalent, the paradoxes obtained on either reading are themselves, at least in this technical sense, equivalent. But we should be interested in much more than merely the generation of paradox. In particular, we should be interested in the various formal connections that hold (or fail to hold) between various paradoxes. In particular, there is a very simple criterion that a 'proper' non-circular, Curry–Yablo hybrid should satisfy – one which is suggested by the aforementioned connection between the Curry paradox and the Liar paradox. Just as the Liar paradox turns out to be merely one instance of the Curry

7 The first such unwinding theorem is proven (for a very simple language) in Cook 2004. Stronger versions of this sort of result can be found in Schlenker 2007a and 2007b.

8 These results already go some ways towards realizing Roy Sorensen's 'General Purge of Self-Reference', which he programmatically describes as follows:

> The simplicity of Yablo's technique invites the conjecture that all . . . paradoxes can be purged of self-reference. The conjecture could be demonstrated if there were a standard formalization of the self-referential puzzles. For one could then formulate an algorithm that mechanically transforms self-referential puzzles into Yabloesque versions. (1998: 150)

The present essay can be seen as further developing this general idea.

paradox – the one obtained by substituting a primitive absurdity constant for Φ – a non-circular Curry–Yablo hybrid should likewise provide us with the Yablo paradox itself when ⊥ is substituted for Φ in its formulation.

Thus, we shall, in the next two sections, briefly construct and examine Yabloesque paradoxes based on (a) and (b), respectively. As we shall see, only one of these paradoxes meets the desideratum of the previous paragraph. Once we have identified the genuine Yablo–Curry hybrid – which we shall call the Yablurry paradox – we shall clear up a few subtleties regarding duality in the appendix.

## 2. *The Dual Yablurry paradox*

The first version of a Curry-style Yabloesque paradox, based on reading (a) above, we shall call the Dual Yablurry paradox, for reasons that will be apparent shortly. Given an arbitrary statement Φ, the Φ-instance of the Dual Yablurry paradox proceeds by considering an infinite $\omega$-sequence of statements of the form:

$S_1$: $((\forall m > 1)(T(<S_{\mathrm{m}}>))) \rightarrow \Phi$
$S_2$: $((\forall m > 2)(T(<S_{\mathrm{m}}>))) \rightarrow \Phi$
$S_3$: $((\forall m > 3)(T(<S_{\mathrm{m}}>))) \rightarrow \Phi$
: : : : :
$S_{\mathrm{i}}$: $((\forall m > i)(T(<S_{\mathrm{m}}>))) \rightarrow \Phi$
: : : : :

– that is, the set of all statements of the form:

$S_n$: $((\forall m > n)(T(<S_m>))) \rightarrow \Phi$

where $n$ is a natural number. We begin by proving that the Φ-instance of the Dual Yablurry paradox entails Φ:

*Proof:* We shall prove by *reductio* that, for any $j$, $S_j$ is true. Assume, for some $i$, that $S_i$ is false. Then (by the relevant $T$-schema):

(1) $\sim(((\forall m > i)(T(<S_m>))) \rightarrow \Phi)$

That is:

(2) $((\forall m > i)(T(<S_m>))) \wedge \sim \Phi$

So:

(3) $(\forall m > i)(T(<S_m>))$

And:

(4) $T(<S_{i+1}>)$

Thus (again, by the relevant T-schema):

$$(5) \quad ((\forall m > i+1)(T(<S_m>))) \rightarrow \Phi$$

But (5) implies:

$$(6) \quad ((\forall m > i)(T(<S_m>))) \rightarrow \Phi$$

This contradicts (2), however. Thus, since $i$ was arbitrary, we conclude that for all $j$, $S_j$ is true – that is:

$$(7) \quad (\forall m > 1)(T(<S_m>))$$

Since $S_1$ is thereby true, we obtain (again, by the relevant $T$-schema):

$$(8) \quad ((\forall m > 1)(T(<S_m>))) \rightarrow \Phi$$

Applying modus ponens to (7) and (8), gives us the desired result:

$$(9) \quad \Phi$$

Q.E.D. [9]

Before moving on, it is worth noting that the Dual Yablurry paradox is not new. It is, in its essentials, just the 'modal-free Yabloesque paradox' presented by Beall (1999) (and the proof that the Φ-instance of the Dual Yablurry entails Φ is similar to Beall's proof). Beall states that:

> ... it would be nice to make a modal-free Yabloesque Curry. Fortunately, a recipe is close at hand. (1999: 738)

Beall does not note that there is more than one way to 'transform' the directly self-referential Curry paradox into an infinite, non-circular Yabloesque construction. As a result, it is somewhat odd that Beall, in searching for a Yabloesque version of the Curry paradox, arrives, not at a genuine Curry–Yablo hybrid, but rather a generalization of the *Dual* of Yablo's Paradox.[10]

9 As before (see n. 6), warnings regarding the use of a metatheoretic $\omega$-rule apply here. As before, see Ketland 2005 for details.

10 The Dual of the Yablo paradox is the infinite $\omega$-sequence of statements of the form:

$S_1$: $(\exists m > 1)(\sim T(<S_m>))$
$S_2$: $(\exists m > 2)(\sim T(<S_m>))$
$S_3$: $(\exists m > 3)(\sim T(<S_m>))$
: : : : :
$S_i$: $(\exists m > i)(\sim T(<S_m>))$
: : : : :

– that is, the set of all statements of the form:

$S_n$: $(\exists m > n)(\sim T(<S_m>))$

where $n$ is a natural number. The Dual of the Yablo paradox and puzzles equivalent, or similar, to it are extensively discussed in Sorensen 1998.

Recalling the desideratum suggested earlier: If we again treat negation as defined in terms of the conditional and a primitive absurdity constant '⊥', then the result of substituting ⊥ for Φ in a Curry–Yablo hybrid ought to result in the Yablo paradox itself (just as substituting ⊥ in for Φ in the original Curry paradox gives us the Liar). The relevant instance of the Dual Yablurry paradox does not provide us with the Yablo paradox, however. Replacing Φ with ⊥ (and treating negation as defined) each line of the Dual Yablurry paradox becomes:

$S_n$: $\sim(\forall m > n)(T(<S_m>))$

which is equivalent to:

$S_n$: $(\exists m > n)(\sim T(<S_m>))$

This sequence of statements, however, is not the Yablo paradox but the Dual of the Yablo paradox. It is for this reason that we have termed the present construction the *Dual* of the Yablurry paradox. Thus, although Beall 1999 contains something that is both Curry-like and Yablo-like (to an extent), the construction he provides is not ideal, in so far as it does not provide us with a genuine Curry–Yablo hybrid. Of course, Beall quite clearly did not mean to be identifying the sole instance of a Curry-like, Yablo-like construction, as is evidenced by his consistent usage of the indefinite 'a modal-free Yabloesque Curry' rather than the definite 'the modal-free Yabloesque Curry'. Nevertheless, as we shall see in the next section, we can do better.

## 3. *The Yablurry paradox*

The second version of a Curry-style Yabloesque paradox, based on reading (b) above, we shall call the Yablurry paradox *simpliciter*. Given an arbitrary statement Φ, the Φ-instance of the Yablurry paradox proceeds by considering an infinite $\omega$-sequence of statements of the form:

$S_1$: $(\forall m > 1)(T(<S_m>) \rightarrow \Phi)$
$S_2$: $(\forall m > 2)(T(<S_m>) \rightarrow \Phi)$
$S_3$: $(\forall m > 3)(T(<S_m>) \rightarrow \Phi)$
: : : : :
$S_i$: $(\forall m > i)(T(<S_m>) \rightarrow \Phi)$
: : : : :

– that is, the set of all statements of the form:

$S_n$: $(\forall m > n)(T(<S_m>) \rightarrow \Phi)$

where $n$ is a natural number. Note that the only difference between the Yablurry paradox and the Dual Yablurry paradox is the scope of the universal quantifier. As in the previous section, we shall first prove that the $\Phi$-instance of the Yablurry paradox entails $\Phi$:

*Proof:* We shall prove the theorem in two parts. First, we shall show that there is at least one $j$ such that $S_j$ is true. Assume, for some $i$, that $S_i$ is false. Then (by the relevant $T$-schema) we have:

(a) $\sim(\forall m > i)(T(<S_m>) \rightarrow \Phi)$

which is logically equivalent to:

(b) $(\exists m > i)(T(<S_m>) \wedge \sim\Phi)$

which entails:

(c) $(\exists m > i)(T(<S_m>))$

This is enough to show that some statement in the list must be true. Now, let $S_j$ be such a true statement. Then, (by the relevant instance of the $T$-schema) we have:

(d) $(\forall m > j)(T(<S_m>) \rightarrow \Phi)$

This gives us:

(e) $T(<S_{j+1}>) \rightarrow \Phi$

(d) also implies:

(f) $(m > j + 1)(T(<S_m>) \rightarrow \Phi)$

which (by the relevant $T$-schema) is just:

(g) $T(<S_{j+2}>)$

Applying modus ponens to (e) and (g) gives us the desired result:

(h) $\Phi$

Q.E.D. [11]

In addition, if we treat negation as defined in terms of the conditional and a primitive absurdity constant '$\perp$', then the original Yablo paradox is just the $\perp$-instance of the Yablurry paradox. Thus, the Yablurry paradox (and not its Dual form suggested by Beall (1999)) meets the desideratum set out at the close of the first section, and thus it (and only it) provides us with a true Curry–Yablo hybrid.

## *Appendix*

The reader might wonder about our calling the paradox constructed in § 2 above the Dual Yablurry paradox. After all, the Dual of the Yablo paradox

11 See the warning in n. 6 and n. 9.

is obtained by replacing each universal quantifier occurring in the relevant set of statements with an existential quantifier. Thus, shouldn't the Dual of the Yablurry paradox be the paradox obtained by replacing each universal quantifier in the Yablurry paradox with an existential quantifier as well?

We need not worry, however. On the 'quantifier-switch' understanding of 'Dual' (which we shall henceforth call Dual*), the Dual* Yablurry paradox would be:

$$\begin{array}{ll} S_1: & (\exists m > 1)(T(<S_m>) \rightarrow \Phi) \\ S_2: & (\exists m > 2)(T(<S_m>) \rightarrow \Phi) \\ S_3: & (\exists m > 3)(T(<S_m>) \rightarrow \Phi) \\ \vdots & \vdots \\ S_i: & (\exists m > i)(T(<S_m>) \rightarrow \Phi) \\ \vdots & \vdots \end{array}$$

– that is, the set of all statements of the form:

$$S_n: (\exists m > n)(T(<S_m>) \rightarrow \Phi)$$

where $n$ is a natural number. We need only recall the fact that:

$$\exists x(\Psi x \rightarrow \Omega)$$

and:

$$(\forall x(\Psi x)) \rightarrow \Omega$$

are logically equivalent in first-order logic in order to see that the Dual* Yablurry and the Dual Yablurry are merely notational variants of one another. Furthermore, the fact that:

$$(\exists x(\Psi x)) \rightarrow \Omega$$

and:

$$\forall x(\Psi x \rightarrow \Omega)$$

are logically equivalent is enough to demonstrate that the Dual* of the Dual Yablurry paradox is merely a notational variant of the Yablurry paradox itself, as we would expect.

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# *A problem for conservatism*

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President Anheuser W. Bush was having a bad day. His latest veto had been overturned, the economy was still in a slump, and his ratings were way down. But that was nothing compared to the problem on his desk right now. It was a doozy, and as usual the French were to blame. Even in the year 2076, with its stupendous technology and permanent conservative majority, this problem wasn't going to go away. It had nowhere to go. Here it was, the eve of the American Tercentennial, and America had no place to celebrate it, because the last remaining piece of American soil had been sold off! How could this have happened? How would history remember him?

Oh, his administration had enjoyed some glorious moments, especially the constitutional amendments: banning flag-burning, banning same-sex marriage, banning progressive income tax, and pulling the USA out of the UN. These were the crowning achievements of the Conservative Permanent Revolution or 'CPR' as it had become known. ('America is dying! It needs CPR!' read a popular bumper sticker.) But there was no need to be vain: these achievements had been built on those of earlier administrations. First, there'd been the deregulation of industry, banking and commerce. Next, there'd been the privatization of all public education. Then those inefficient national parks had been sold off to the Disney Corporation and various oil, paper and

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