CIRCULARITY AND INFINITE LIAR-LIKE PARADOXES

By

JESSE BUTLER

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Jesse Butler

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Chair: Greg Ray
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Before Stephen Yablo's "Paradox without self-reference" it was commonly held that what was loosely and vaguely termed "self-reference" was necessary for semantic paradoxes of the liar-like variety. Yablo claimed that the infinite, liar-like set of sentences he provided proved otherwise. In subsequent years, Beall, Bueno and Colyvan, Priest, Sorensen and Tennant (and others) have debated whether Yablo has in fact demonstrated that self-reference is not required for paradox by exhibiting a paradoxical, yet completely non-circular set of sentences. My aim in this paper is two-fold. First, I try to clarify this debate over whether Yablo's set of sentences is self-referential (in the same way that the traditional liar sentence is) as preparation for the assertion that it is not. I argue that the situation regarding sets of sentences that form candidates for liar-like paradoxes can be plausibly formalized using directed graphs and that this treatment bears out my assertion. Second, I address Priest's claim that even given the lack of this kind of self-reference, Yablo's sentences are in *some* sense self-referential because they, in Priest's somewhat mysterious terminology, "concern a circular predicate," as do,

according to him, all standard semantic and set-theoretic paradoxes. I argue that if a set of sentence concerns a circular predicate, there is a kind of recipe to construct the sentences of that set. Even though the word “recipe” is still a bit too metaphorical to do much good, another formal treatment is available that captures this intuitive notion and yields a sufficient condition for the paradoxicality of infinite sets of liar-like sentences. Using this formal treatment, it is easy to show that there must be an infinite set of liar-like sentences that is not constructible from a recipe, and so in Priest’s terms does *not* concern a circular predicate. Thus, while Yablo’s paradoxical set of sentences is circular in Priest’s sense, this is incidental. We will have proved that there is a variant of Yablo’s example that is not circular in either the narrow referential sense nor in Priest’s broader sense.

CHAPTER 1
YABLO'S PARADOX IS *INFERENTIALLY CIRCULAR* BUT NOT *REFERENTIALLY CIRCULAR*

Stephen Yablo (1993)[1] billed his paradox (refered to as "Yablo's Paradox" or "*YP*" hereafter) as a demonstration that self-reference was not *necessary* for liar-like paradoxes.[2] Of course, philosophers have argued that *YP* either is or is not self-referential and have so argued as to whether it fails or succeeds as a demonstration that self-reference need not be involved in paradox. My first goal in this paper is to get clear about several of the arguments for and against *YP*'s self-referential (or to speak more generally *indirectly self-referential* or *circular*[3]) character. In sorting out whether *YP* is or is not circular, and if so in what way, I assert that we can understand more generally the phenomenon of liar-like paradoxes comprising infinite sets of sentences if we try to "picture" graph theoretically the references made by each sentence. With this technique, we see that none of the sentences of *YP* are indirectly self-referential with the references they make. So *YP* is not referentially circular in the way traditional finite liars are. The second part of the paper attempts to show that there is a variant of Yablo's Paradox that does not concern a predicate of the natural numbers with circular satisfaction conditions,

1 For all references to Yablo see (Yablo 1993).

2 I take liar-like paradoxes to comprise sets of sentences each of which only makes truth claims about itself or the other sentences.

3 The idea being that even though the two sentences: "The second sentence enclosed in double quotes on page 1 is true," and "The first sentence enclosed in double quotes on page 1 is untrue," are not *themselves* self-referential (each makes no claim about itself) they exhibit indirect self-reference. The first makes a claim of the second which in turn makes a claim about the first. "Circularity" might describe this more general notion. Occasionally, I will use that term to speak of this general form of indirect self-reference.

and so is not circular in the sense that Priest (1997)[4] claims *YP* is. I will argue that an infinite list of sentences can concern a circular predicate only if there is a recipe to generate the sentences of the list. We do this by representing each sentence of the infinite, supposedly paradoxical list as a recursive function from a subset of the natural numbers to {0, 1}, and placing minimal conditions on the behavior of each of those functions relative to others of the "family" of functions that represent the sentences of the infinite list. Once we have represented infinite lists of sentences that are paradoxical in this way, it is easy to see that the number of such lists is uncountable. The number of recipes for infinite lists is countable, so the paradoxial infinite liars must outstrip the paradoxical infinite liars that concern predicates with circular satisfaction conditions.

We begin (§1.1) with a short rehearsal of *YP* by exhibiting the infinite list of sentences that form the paradox, and (§1.2) a short demonstration that a most general form of circularity (inferential circularity) must be involved in any sort of liar-like paradox. In §1.3, I will try to recreate the dialectic provided by the arguments of Tennant (1995)[5], Priest, Sorenson (1998)[6], Beall (2001)[7] and Bueno & Colyvan (2003)[8] regarding *YP*'s self-referential character. In §1.4, I will consider Tennant's argument in a bit more detail and elaborate on its shortcomings while tentatively endorsing his conclusion. In that section, I will propose what I believe to be a better method for demonstrating that *YP* is not even indirectly referentially circular. By using this method, we will be able to

[4] For all references to Priest see (Priest 1997).

[5] For all references to Tennant see (Tennant 1995).

[6] For all references to Sorensen see (Sorensen 1998).

[7] For all references to Beall see (Beall 2001).

[8] For all references to Bueno and Colyvan see (Bueno and Colyvan 2003).

more easily separate the issues of what Sorensen calls, "self-reference at the level of specification and [self-reference] at the level of content," and that this separation will guide us to clearer thinking about the present problem.

In §2, I will consider in detail parts of Priest's paper, while explicating and assessing his argument, I propose that his observations may be reformulated in a more productive treatment of the general problem. This will lead to a method of formalizing, in some generality, liar-like paradoxes comprising infinite lists of sentences. In these terms, we will be able to see that Priest's charge that *YP* "concerns a circular predicate" can be reframed as a observation that there exists a certain recipe for constructing the sentences of *YP*. Once we have seen that both Priest's and Beall's argument that *YP* is circular stem from the fact that its the sentences can be so generated, and armed with the formal characterization of paradoxicality in terms of infinite liar-like sets (§2.2), we will be in position to show (in §3) that there must be a "Yabloesque" paradox, the sentences of which cannot be generated by a recipe.[9] Thus, even if Priest is right that *YP* is circular in a broad sense, we will have shown that such circularity is not essential to Yabloesque paradoxes.

1.1 Rehearsing YP

Yablo's Paradox concerns a countably infinite list of sentences s_0, s_1, s_2, . . . of the following sort:

s_0: for all $k > 0$, s_k is untrue.
s_1: for all $k > 1$, s_k is untrue.
. . .

[9] We cannot *precisely* describe it as it contains non-constructive elements.

These sentences clearly form a paradox in the sense that they have no stable assignment of truth-values. If s_0 is true then each of s_k for $k > 0$ is untrue, so then each of s_k for $k > 1$ is untrue, but in this case, s_1 is true after all because that's just what it claims. So s_0 cannot be true because it claimed that s_1 was untrue. On the other hand, if each of $s_0, s_1, s_2, \ldots$ is untrue then in particular each of $s_1, s_2, \ldots$ is untrue, and since the latter circumstance is precisely the condition under which s_0 is true, s_0 must be true after all. For future reference, let the set S be this collection of Yablo sentences, $\{ s_0, s_1, s_2, \ldots \}$.

1.2 Inferential Circularity

Before addressing *YP* in more detail, let us notice a general point. As soon as we begin considering liar-like paradoxes, we notice that there is always some element of what we would commonly term "circularity" involved. More specifically, it is clear that there must be a certain logical relationship between sentences comprising a paradox in light of which we can make an assumption about the truth-value of a certain sentence, draw inferences about other sentences, then return to (as if we had completed a kind of reasoning *circle* or *loop*, but not engaged in *circular reasoning*) and reassess the truth-value of the original sentence. Insofar as any truth valuation of sentences of a liar-like paradox leads to inconsistency and we can demonstrate this inconsistency, it must be the case that those sentences bear certain logical relations to each other that allow for this demonstration. I want to call sets of sentences the members of which bear these logical relationships "inferentially circular" in light of the fact that these sort of sets enable us to draw inferences which are circular in so far as these inferences begin and end with assertions about truth value of the same sentence. For example, if we reason about a "chain" set of liar sentences and see that the set has no truth valuation, we go through a process resembling this:

> If the (say) i^{th} sentence is true (or untrue), then $i+1^{\text{st}}$ sentence is true (untrue), but this means that the $i+2^{\text{nd}}$ is true (untrue), . . . , but in that case the i^{th} sentence must have been untrue. On the other hand, if the i^{th} sentence is untrue, then the $i+1^{\text{st}}$ is untrue (true) and the $i+2^{\text{nd}}$ is untrue (true), and . . . , but in that case the i^{th} sentence must have been true.

In order for inconsistency to result from each assumed truth-value of the i^{th} sentence, each of the sentences must bear logical relations to the accompanying sentences. In the case of liar-like paradoxes, sentences make only truth claims of each other, and so it is *only* such truth claims that provide the logical relationships which lead to inconsistency and paradox. And so every set of liar-like sentences that we claim to be demonstrably paradoxical must, in virtue of that claim, be inferentially circular. *YP* provides no exception: it must be inferentially circular because we can reason to contradiction from the truth of any of its sentences. In any of the arguments I consider or offer myself, I assume that assertions that *YP* is or is not circular *are not* assertions about whether it is or is not inferentially circular. Inferential circularity is essential to any paradox, *YP* included.

For the liar-like paradoxes that came before *YP*, inferential circularity was typically grounded in some kind of *content* (what we will call later *referential*) circularity. That *YP* has the requisite inferential circularity raises a justifiable suspicion that is must also feature some sort of underlying content circularity. This question is one target of our investigation. Another target is to determine how *YP* is *predicatively circular* and whether there other similar sets of sentences that are paradoxical yet are not *predicatively circular*. This locution comes from Priest who claims that *YP*, although not circular in exactly the same way the well-known liar sentence, *is* circular in that it "concerns" a circular predicate of the natural numbers and that use of this particular predicate is in some way crucial to understanding the logic of the paradox. The concerned predicate is circular in that its satisfaction by a certain number, *n*, depends upon whether it is satisfied by any natural number greater than *n*. The predicate's satisfaction conditions can be stated only in terms of the predicate, that is, only *circularly*.

1.3 Recreating the Dialectic

The goal of this section is to try to recreate a particular dialectic surrounding *YP* and circularity. In laying out the dialectic it is important to notice first that often the participants talk past each other. Tennant claims that *YP* is not *at all* self-referential, Graham Priest begins to mark a distinction between two different kind of circularity when he says, "Note that each sentence refers to (quantifies over) only sentences later in the sequence. No sentence therefore refers to itself, even in indirect, loop-like fashion" (237). But then goes on to claim on the page that, "[his formal diagnosis of *YP*] show that we have a fixed point . . . of exactly the same self-referential kind as in the liar paradox . . . The circularity [of *YP*] is not manifest." Priest's claim is that *YP* merely concerns a circular predicate, and that the predicate's involvement (rather than indirect self-reference) is the source of the paradox's inferential circularity. Much of the foregoing disagreement results because they do not appreciate the distinction between the two different kinds of circularity. To get a handle on things, we need first to get clear on what Tennant's claim is in light of the fact that inferential circularity seems necessary for any sort of liar-like paradox, and second to mark a further distinction in kinds of circularity to help clarify the conversation between Priest, Sorensen, Beall and Bueno and Colyvan.

To the first end, if we compare a crude version of a *pair liar* (see footnote 3) with *YP*, we're immediately struck by the difference: the pair liar contains sentences that make truth claims of themselves (albeit indirectly), whereas the sentences of *YP* do not seem to do this. These attempts to capture and explain the difference between these two sorts of paradox account for one strain of a dialectic concerning *YP*. One example I have in mind is Neil Tennant's. If Tennant would agree to our claim that all liar-like paradoxes are inferentially circular (and it seems that he would), we might attribute to him, instead of his more radical pronouncement above, the claim that *YP*'s inferential circular is not rooted in any sort of content circularity, as is the inferential circularity of any finite liar. And so we can read one of Tennant's conclusions as being that inferential circularity need not be rooted in content circularity. I endorse this conclusion and will say more about it shortly.

As far as the second issue goes, let us stipulate two further different kinds of circularity. Say that a set of sentences is *referentially circular* if a particular sentence of the set is either directly self-referential (like the traditional liar sentence) or *indirectly self-referential*. We can explain indirect self-reference by making precise what a sentence of a liar-like set, which is not directly self-referential, refers to. Such sentence (call it *p*) makes use of restricted quantification to pick out a set (*Q*) of sentences (which is a proper subset of the set to which *p* belongs because *p* is not directly self-referential) and then makes some truth claim of some members of *Q*. For instance, if the sentences are organized in an ordered list like those of the *S* of *YP*, then the sentence *p* might be something like, "Some of the following sentences are untrue," and so *Q* would be the set of all the sentences that came after *p* in the list. The sentence *p directly refers* to all the sentences of *Q*. If there is a sentence, *q* (a member of *Q*) that directly refers to a set of sentences *P*, which includes *p*, then we can say that *p* refers indirectly to itself (is indirectly self-referential) via the intermediary *q*. And so generally, we say that a set of sentences is indirectly self-referential if a member of the set refers indirectly to itself via any number of intermediaries. For example, the pair liar sentences of footnote 3 clearly form a set that is referentially circular.

On the other hand, a set of sentences that, in Priest's terms, concerns a circular predicate is, as we mentioned earlier, predicatively circular.[10] With these terms in mind, we see that Priest claims that *YP* is predicatively circular but not referentially circular. We have seen that every liar-like paradox must be inferentially circular, Tennant has argued that referential circularity is not required for paradox (and more generally inferential circularity), and Priest has hinted that referential circularity is not required for predicative circularity. We still lack a detailed argument for independence of referential and predicative circularity, and whether every liar-like paradox is predicatively circular is an open question.

If the distinction makes clearer Priest's claim, it is a bit surprising when Sorensen weighs in on *YP* that he begins to muddy the waters a bit (145-149). Sorensen's intuition is that self-reference at the "level of content" (i.e. referential circularity) can be separated from self-reference (or circularity) at the "level of specification" (i.e. predicative circularity). It seems that Sorensen thinks, as does Priest, that *YP* is not referentially circular but is predicatively circular, and thinks that predicative circularity can feature in paradoxes the sentences of which are not referentially circular, all of which makes the following claim seem very odd: "Priest is committed to saying that that the infinite, indirect liar is doubly self-referential – once at the level of specification, again at the level of content" (148). Whether or not Sorensen thinks something in Priest's method commits

[10] Priest's locution seems a bit mysterious. In §2, I try to fill out a bit of what Priest may intend.

him to identifying referential and predicative circularity, the above quote does say that Priest claims that *YP* is both referentially and predicatively circular. Whatever the reason for Sorensen's misreading of Priest, a confusion begins here which, I believe, runs the whole dialectic off-track. Beall responds to Sorensen's claim that *YP*'s predicative circularity has no bearing on its referential circularity. In the following quote Beall seems to argue that predicatively circular sets of sentences must also be referentially circular, and cites Priest's demonstration of YP predicative circularity as evidence for its referential circularity (my emphasis; it is meant to show where Beall hints that predicative circularity is not separable from referential circularity).

> If the foregoing is correct, then the issue becomes clear: if we have fixed the reference of 'Yablo's paradox' at all, then we have fixed the reference of 'Yablo's paradox' via (attributive) description. But, now, the upshot of Priest's point is plain: Priest has shown that any description *we* employ to pick out (or otherwise define) a Yabloesque sequence is circular; this much Sorensen concedes. *From here, however, it is a small step to the circularity of the sequence itself.* We are fixing the reference of 'Yablo's paradox' via (attributive) description, which means that 'Yablo's paradox' denotes whatever satisfies the given reference fixing description. The situation, however, is this: that the satisfaction conditions of our available reference-fixing descriptions require a circular satisfier – a sequence that involves circularity, self-reference, a fixed point. *Given all this, it follows that the reference of 'Yablo's paradox' is circular*. . . (180)

Finally, in response to Beall, Bueno and Colyvan argue that there are denoting terms (like "the integers" or "the *supremum* of set *A*") whose referents can only be fixed by the use of a circular (recursive) description but whose referents are not themselves circular in any sort of obvious way. According to them, the situation with *YP* is analogous.

I think that the tedious attention to whether reference to *S* can be fixed by demonstration plus description or by description alone is misplaced. We could have avoided this problem by paying closer attention to Priest's original paper (or ignoring

Sorensen's misreading of it), and by using a more intuitive explanation of exactly why the set of sentences that form *YP* is not referentially circular.

1.4 Characterizing Referential Circularity with Directed Graphs

We want to show ultimately that referential circularity is entirely independent of predicative circularity. A first step toward that is to take a position with regard to the dialectic I have just presented. Specifically, I want to argue that *even if YP* is predicatively circular it is *not* referentially circular. To this end, we will develop a general method for detecting referential circularity independent of predicative circularity, and use it to show that *YP* is not referentially circular. But before doing so I feel I need to justify the more specific goal. My worry is that Tennant has already accomplished that which I have just articulated, and if only we adverted to his method and conclusion, there would be no need to pursue *another* way of seeing that *YP* is not referentially circular. In response to this concern, I will briefly summarize a relevant part of Tennant's paper and then say why his method is not suited for the purposes of the present work.

Tennant believes that he's provided a proof-theoretic diagnosis of paradoxicality in previous work which makes self-reference neither necessary nor sufficient for paradox (of the liar sort), and sees *YP* as a test for his account (199). One conclusion he reaches is that, "[*YP*'s] apparent lack of self-reference . . . is reflected, on my account in the non-looping character of the reduction sequence . . ." (206). Toward the end, he claims that his method yields a condition for self-reference of liar-type sentences, "I shall make so bold as to suggest that it is precisely when the non-terminating reduction procedures enter loops that self-reference is involved [in the paradoxicality of the set of sentences in question] . And when they do not enter loops - as with Yablo's example - then self-reference is not involved" (207). Proofs of absurdity in his system from traditional liars do enter loops and thus are self-referential.

These conclusions can be drawn because appropriate proofs of absurdity are available. It is the examination of the structure of proofs from different sets of sentences that allow us to detect self-reference. His paper is concerned specifically with *YP*, so he makes use of two "*id est*" inference rules: first, from TS_n infer $(\forall k{>}n)\neg TS_k$, and, second, from $(\forall k{>}n)\neg TS_k$ infer TS_n (203). Since there is a inference rule for each sentence, the proof theoretic procedure is dependent upon knowing (truth functional equivalents of) the truth claims of each sentence of the set.

What we should notice is that this sort of proof theoretic approach to detect referential circularity seems to require that a set of sentences be predicatively circular. To show that contradiction results from the supposed truth of any particular sentence, Tennant's proof of absurdity make use of the precise truth claims of each sentence of the set. At least in the kind of proofs that Tennant presents, the *id est* inference rules require that the absolute (non-relative) truth claims are made by each and every sentence are available to us. For an infinite set, to know these absolute truth claims, we must have a recipe for what the sentence of the set are or at least for their absolute truth conditions. As foreshadowing, I assert that a set of sentences is predicatively circular only if there is some sort of "recipe" to construct each of the sentences of the set, so a set cannot be predicatively circular if there is no recipe that tells precisely what a sentence of a particular index claims. On the face of things, it seems that Tennant's method for showing circularity depends upon the *id est* inference rules, or at least on predicatively circularity. Since we want to show that *YP* is not referentially circular independently of its predicative circularity, there is at least *prima facie* motivation for us to search for another way to see this. In sum, I agree with Tennant's conclusion and do not want to point out any mistake in his reasoning[11], but rather to suggest that we need a different method for detecting referential circularity that is not bound quite so closely to the predicative circularity of a set of sentences.

To begin, recall that all such sets liar-like set that are paradoxical are inferentially circular. So in any finite set, we can always "start" at some particular sentence (say *i*) and by "following" the direct references work our way back to *i*, so that we see that sentence *i* was indirectly self-referential. Intuitively, this is just what referential circularity comes to. If we could somehow "picture" the referential structure of the paradox independently whether the set of sentences concerns any circular predicate, and instead make use only of how each sentences restrictedly quantified over others as an initial step toward making truth claims about those sentences, and then use a specific feature of this picture to assess whether the sentences of the paradox were referentially circular, we would have another way to show that referential circularity was independent from predicative circularity. This method for detecting referential circularity would not be so closely bound to any predicate that the set concerns. So in order to show that *YP* lacks referential circularity with appeal only to the restricted quantification presented by each of it is sentences[12] (rather than explicit appeal to it is particular predicate that the set

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11 There is, however, a problem with Tennant's presentation; when he begins the analysis of *YP* in terms of his formal system, he renders generic members of *S* "formally and uncontroversially as follows: S_n: $\forall k>n\ \neg T(S_k)$" (203). Greg Ray points out that Tennant makes a use / mention mistake. "s_n" is the *name* of a sentence of S, so the "*n*" in the preceding quoted string is a *substring* – a constituent of that name. It does not make sense to quantify over such a thing. Priest addresses this issue, and I discuss his presentation in the next chapter.

12 Appeal just to this restricted quantification does not hurt our chances for demonstrating a set of sentences not constructible from a recipe because, to foreshadow again, the size of the set of lists of

concerns), I develop a graph theoretic test for referential circularity of an arbitrary (finite or infinite) list of sentences that is a candidate for being a liar-like paradoxical set.

As a preliminary, a bit about *directed graphs* (*digraphs*). They can be visualized as geometric figures consisting of *vertices* ($\boldsymbol{V_1}$, $\boldsymbol{V_2}$, . . .) and directed *edges* ($\boldsymbol{E_1}$, $\boldsymbol{E_2}$, . . .) which connect pairs of vertices by running *from* one *to* the other. A *path* through a graph is a vertex, edge, vertex, edge, . . . edge, vertex sequence such that if "$\boldsymbol{V_i}$, $\boldsymbol{E_j}$, $\boldsymbol{V_k}$" is an arbitrary portion of the sequence (where possibly, $i = k$), $\boldsymbol{E_j}$ is an edge *from* $\boldsymbol{V_i}$ *to* $\boldsymbol{V_k}$. A *cycle* is a path that has the same first and last vertex.

From a set of sentences that is a paradox candidate, we can construct a directed graph ($\boldsymbol{G}$) that will have the same number ($\boldsymbol{n}$) of vertices as the candidate has sentences ($1 \leq \boldsymbol{n} \leq \omega$). Vertex $\boldsymbol{V_i}$ of $\boldsymbol{G}$ will "correspond" to the i^{th} sentence of the set. If the i^{th} sentence refers or quantifies over the j^{th}, k^{th}, . . . , l^{th}, . . . sentences in the truth claims it makes, draw an edge from $\boldsymbol{V_i}$ to each of $\boldsymbol{V_j}$, $\boldsymbol{V_k}$, . . . , $\boldsymbol{V_l}$, . . . [13] The list of sentences is *graphically circular* iff its corresponding directed graph, $\boldsymbol{G}$, has a cycle.

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sentences each of whom restrictedly quantifies over all those who follow it outstrips the size of the set of lists of sentences that are recursively constructible (alternatively: "concerning a circular predicates"). We will find one that is a member of the first set and not a member of the second. And then, we will be almost all the way to finding a referentially circular set of sentences that is not predicatively circular.

13 The intuitive idea is to connect with edges all those vertices which correspond to the sentences of which the i^{th} sentence restrictedly quantifies over in order to makes truth claims. At first, it may be troubling that two sets comprising dissimilar sentences may have exactly the same corresponding digraph; two sets of sentences will have the same corresponding digraph exactly when corresponding sentence (sentences with the same index) of the respective sets restrictedly quantify over the same sets. For example, consider two infinite sequences of sentences (S' and S'') as candidates for a liar-like paradox. The first comprises only sentences of the form, "s'_i: at least two of the following are untrue," and the second comprises only sentences of the form, "s''_i: at least forty-eight of the following are untrue." Every sentence of S' and S'' quantifies over all the sentences that come "after it" in S' or S''. And so, the digraphs corresponding to each of the candidates will have the same edge / vertex structure, i.e. Edges from V_1 to each of V_2, V_3, V_4, . . . , edges from V_2 to each of V_3, V_4, V_5, . . . That the corresponding graphs $G_{S'}$ and $G_{S''}$ have exactly the same structure is not troublesome because, to consider the question most generally, we realize after some consideration that the truth conditions of s'_i and s''_i depend upon *any* of the sentences of S' or S'' respectively that have index higher that i.

Graphical circularity of a set of sentences guarantees that the corresponding graph $\boldsymbol{G}$ is "circular" in the sense that we could traverse $\boldsymbol{G}$ (starting and ending at some $\boldsymbol{V_i}$) if we were allowed to "step" from one vertex to another iff there was an edge from the former to the latter. Since the vertex / edge structure of $\boldsymbol{G}$ mimics the truth claim structure of the corresponding set of sentences at least one of the sentences of the graphically circular set must be directly or indirectly self-referential. And so I claim that a set of sentences of a candidate for a liar-like paradox is referentially circular iff the corresponding graph is graphically circular.[14] Of course, it is possible for a set of sentences to be referentially circular and not form a traditionally paradoxical set (for example a truth teller). If a detection method for referentially circular paradoxicality is wanted, it is easily accomplished by considering algorithms for *coloring* our directed graph in an appropriate way. I hope that this technique for detecting paradoxes will go some way toward vindicating our characterization of referentially circular sentences in terms of corresponding graphically circular graphs. We provide a formal method for so deciding when a candidate forms a paradox in *Appendix A*.

As expected, the set of sentences that constitute *YP* turn out not referentially circular on our characterization. If we form the associated graph and call it $\boldsymbol{G_{YP}}$, notice that for each i, all the edges from $\boldsymbol{V_i}$ (edges which mimic the truth claims of the i^{th} sentence) go to only vertices with index greater than $\boldsymbol{i}$, so it is impossible for $\boldsymbol{G_{YP}}$ to contain a cycle and so impossible for it to be graphically circular.

14 Something else to notice is that if we require that a sentence from language L_1 can make truth claims about a sentence from L_2 iff L_1 is a metalanguage for L_2, then sets of sentences whose corresponding graphs are circular are disallowed. If the 1st sentence claims something about the truth of the 2nd sentence, the 2nd about the 3rd, then if we're observing the metalanguage / object language distinction, no sentence is allowed to make truth claims of a sentence whose number is less than the number of the former. It is just this (forbidden) kind of structure that allowed for circular digraphs.

CHAPTER 2
TO CONCERN A CIRCULAR PREDICATE IS TO BE RECURSIVELY CONSTRUCTIBLE

I have argued that a simple-minded graph theoretic technique shows us that *YP* is not referentially circular, and so have endorsed a claim made by Priest. I turn now to his assertion that *YP* is in some sense circular because it concerns a predicate with circular satisfaction conditions (in my terminology *predicatively circular*). I think the intuition is right – *YP* does in some sense concern a predicate with circular satisfaction conditions. I also think that this is not an essential feature of Yablo-type paradoxes, but to see this we must first get clear about Priest's argument.

2.1 Graham Priest's Argument Considered More Carefully

Recall that the sentences of *S* clearly form a paradox in the sense that they have no stable assignment of truth-values, and we can see this because we can run through the reasoning like that presented in §1.1. Graham Priest maintains that *YP* involves self-reference, but is not referentially circular.

> Formalizing the sentences with a truth predicate, *T*, we have that for all natural numbers, n, s_n is the sentence $\forall k > n, \neg Ts_k$. Note that each sentence refers to (quantifies over) only sentences later in the sequence. No sentence, therefore, refers to itself, even in an indirect, loop-like, fashion (237).

He goes on to argue that *YP* concerns a circular predicate. He begins by proposing a formalization of the reasoning to contradiction from an arbitrary sentence of *S*,

1. $Ts_n \Leftrightarrow (\forall k > n), \neg Ts_k$
2. $\Rightarrow \neg Ts_{n+1}$
3. $Ts_n \Leftrightarrow (\forall k > n), \neg Ts_k$

4. $(\forall k > n+1), \neg Ts_k$
5. $\Rightarrow Ts_{n+1}$
6. ✕ (from (2) and (5))
7. $\Rightarrow \neg Ts_n$
8. $\Rightarrow (\forall k) \neg Ts_k$
9. $\Rightarrow (\forall k > 0) \neg Ts_k$
10. $\Rightarrow Ts_0$
11. ✕ (from (8) and (10)).

Where (8) must be justified by something akin to universal generalization for Priest's semi-formal scheme, since the "*n*" of the previous lines was arbitrary.

The observation is that since we generalize away the "*n*" in step (8) it must play the role of a variable in the above treatment. In that case, "s_{n+1}" cannot be the name of a sentence. Rather, Priest says, "*s*" must be understood as a name of a predicate which applies to natural numbers. But in that case, we cannot just apply the truth predicate to occurrences of, e.g., "s_{n+1}". We have to make use of the two-place *satisfaction* (SAT) relation between numbers and predicates, and s, a one-place predicate of the natural numbers, and rewrite as follows.[1]

12. $\text{SAT}(n, \mathsf{s}) \Leftrightarrow (\forall k > n), \neg\text{SAT}(k, \mathsf{s})$
13. $\Rightarrow \neg\text{SAT}(n+1, \mathsf{s})$
14. $\text{SAT}(n, \mathsf{s}) \Leftrightarrow (\forall k > n), \neg\text{SAT}(k, \mathsf{s})$
15. $\Rightarrow (\forall k > n+1), \neg\text{SAT}(k, \mathsf{s})$
16. $\Rightarrow \text{SAT}(n+1, \mathsf{s})$
17. ✕ (from (13) and (16))
18. $\Rightarrow \neg\text{SAT}(n, \mathsf{s})$
19. $\Rightarrow (\forall k)\neg\text{SAT}(k, \mathsf{s})$
20. $\Rightarrow (\forall k > 0)\neg\text{SAT}(k, \mathsf{s})$
21. $\Rightarrow \text{SAT}(0, \mathsf{s})$
22. ✕ (from (19) and (21))

1 Priest does not carry out the following derivation but merely suggests what it might look like.

Priest's assessment of the situation is a little hard to follow[2], but we can see that the original formulation began (1) with premise stating the truth conditions of an arbitrary sentence of *S*. The second formulation starts with a statement of the satisfaction conditions of the predicate **s**. We see from (12) plainly that the predicate has circular satisfaction conditions.

I think that something in Priest's claim rings true. In his terms, predicate **s** has a fixed point, and if his treatment succeeds in capturing what turns the trick in *YP*, then there is a sort of circularity involved. One difficulty, however, is to see from his presentation *precisely what role* he thinks predicate **s** plays in *YP*. What does it mean for a set of sentences to "concern" a circular predicate? Priest claims to have uncovered and made explicit the logical form of the sentences of the Yablo sequence, and, in so doing, he's shown that each of these logical equivalents refer to the predicate **s**. So the natural understanding of "concern" is that each sentence of the set mentions that predicate. The set is predicatively *circular* because the predicate so referenced has circular satisfaction conditions.[3]

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[2] Priest claims on 237 that "the fact that **s** = '$(\forall k > x)(\neg \mathrm{SAT}(k, \mathbf{s}))$' shows that we have a fixed point, **s**, here of exactly the same self-referential kind as in the liar paradox. In a nutshell, **s** is the predicate 'no number greater than *x* satisfies this predicate'. The circularity is now manifest." Professor Ray points out that it is a bit misleading to claim that **s *is*** '$(\forall k > x)(\neg \mathrm{SAT}(k, \mathbf{s}))$' because **s** could be *any* predicate which had the *same* satisfaction conditions.

[3] It might also be argued that each sentence of the Yablo sequence *uses* the predicate s, and that a set is predicatively circularity if each member *uses* a predicate with circular satisfaction conditions. In this case, there are arguments exactly parallel to the one in chapter 3 aimed to show that there is a paradoxical sequence that is not predicatively circular.

2.2 A Different Formal Treatment of *YP* and Other Similar Sets of Sentences

We're aiming to show that there is a set of sentences that is paradoxical but which is not predicatively circular. To do this, we will try to show that there is a class of sentences for which there can be no recipe – we cannot tell precisely what the truth conditions for an arbitrary sentence are – but which is clearly paradoxical. If we can show that there is such a class, then we can reason that Priest won't be in a position to assert that each sentence of any member of this class must reference a predicate with circular satisfaction conditions. There are some sentence sets that are paradoxical that needn't be predicatively circular. At first glance, our task may seem impossible: if a set of sentences is not predicatively circular, we cannot know the truth conditions of an arbitrary member of the set. How are we to know that the sequence is paradoxical? The only method we have seen so far for determining paradox is the sort of reasoning in sentences (12)-(22). That treatment made use of a predicate that was mentioned by each sentence of a particular list to give truth conditions in terms of the truth or untruth of each of the subsequent sentences. In this section, we will argue that we can characterize a list of sentences without appeal to a single, certain circular predicate that is mentioned by sentences. If certain restrictions can be placed on the truth conditions set forth by an arbitrary sentence of the set, perhaps only relative to the truth conditions of another sentence, then it is possible to argue that the set forms a paradox without reasoning that makes use of precisely specified truth conditions for an arbitrary sentence. The motivation behind this technique should become clear as we develop the method in the next few pages.

Recall that s_i, the i^{th} sentence of S, is "For all $k > i$, s_k is untrue." If we let **0** go proxy for "untrue", **1** for "true", and in some loose way (to be made clear) associate the output of a function from a subset of the natural numbers to {0,1} at a particular value, n, with the truth claim made about sentence with index n, by a the sentence corresponding to this function, then for each i, f_i imitates s_i, where for each i, f_i has domain $\{y: y > i\}$ and for all $n > i$, $f_i(n) = 0$. Briefly, just as *YP*'s i^{th} sentence claims that each of the succeeding sentences is untrue, each f_i is zero over its whole domain (natural numbers greater than i). Working with these sort of "imitators" is the first step toward our more general characterization of Yabloesque liars.

To capture more of what was at work in *YP*, recall that the first part of its bite came from the fact that, for instance s_0 and s_1 made *exactly the same* truth claims about each sentence whose index was greater than 1, so if the first were true then we could reason that the second would have to be true also, but the first claimed the second was untrue, so, on pain of contradiction, the first must have been untrue after all. Similar reasoning could be repeated for any of the sentences of *S*. The second part of the bite was that one of *S* had to be true, otherwise, since what (say) s_0 claims to be the case would in fact be the case, and s_0 would have been true after all. Another way in which inconsistency might occur in an ordered list sentences which only make truth claims about each other is if sentence x claims that both sentences y and z are true, but y claims that z is untrue.[4] Of course, the last *was not* a feature of the sentences of *YP*. With this in mind, we can create a model of infinite sets of sentences each of which makes only truth claims about those of greater index, for which there is a formal analog of paradoxicality.

[4] There are more ways in which the truth of one of such a list of sentences could lead to inconsistency. For instance, if sentence with index 6 (call it "s_6") claimed that s_7 was true and that s_9 was untrue, that s_7 claimed that s_8 was true and s_8 claimed that s_9 was true. I do not consider this sort of pathology because the aim is to show that if we have a set of sentences which spawn inconsistency only by the two ways mentioned, then the paradoxical sets "outstrip" those which concern a circular predicate. To do this it will be enough to consider the two kinds of pathology just mentioned.

The type of functions we need for this task will be doubly indexed with subscripts and collected into sets with an order induced by their subscripts. I will call the ordered sets into which our functions are gathered "families"[5] from now on for convenience. The constituent functions themselves, like (23) above, are supposed to imitate the ordered, infinite member sentences of a candidate for a liar-like paradox and each family will represent a single candidate for paradox. For example, if such a family is $D_l = \{d_{l,1}, d_{l,2}, d_{l,3} \ldots\}$, then $d_{l,n}$ has as domain the natural numbers greater than n and as range $\{0, 1\}$. The domains of the members of a family are so constrained because the sets of English sentences to be based on these families have a corresponding constraint – each member of the sort of sets we will be considering makes claims only of sentences of greater index. In other words, the sets under consideration are not referentially circular. Recall that the members of the range will be proxies for "true" (1) and "untrue" (0). Call a member function, $d_{l,i}$, *nullish* iff$_{\text{df}}$ for all x in its domain, $d_{l,i}(x) = 0$. Call a member function, $d_{l,i}$, *semi-nullish* iff$_{\text{df}}$ for some $x > i$, for all $y > x$, $d_{l,i}(y) = 0$. Say that member function, $d_{l,i}$, *triangulates* member function $d_{l,j}$ ($j > i$) iff$_{\text{df}}$ there is some $x > j$ such that $d_{l,i}(j) = 1$, and $d_{l,i}(x) \neq d_{l,j}(x)$.

A family of such functions, D_l, is $\mathcal{R}$ iff *each* of the member functions $\{d_{l,1}, d_{l,2}, d_{l,3} \ldots\}$ is recursive and is $\mathcal{P}$ iff the family contains an infinite number of nullish member functions, and for each member function which is not nullish either it is semi-nullish or it

[5] One might be concerned that "family" is a poor choice of terminology for what are essentially *ordered lists* of functions. I think "family" is a better, more descriptive term that is appropriate because an order is induced on a family by the fact that each of its members has a different domain – a function whose domain is a subset of another can be said to follow the latter in order in which functions are collected into families.

triangulates some other member function. Call a family $\mathcal{R}$-$\mathcal{P}$ if it is both $\mathcal{R}$ and $\mathcal{P}$.[6] In these terms, a family each of whose members is nullish imitates *YP*, and such a list is $\mathcal{R}$-$\mathcal{P}$.

> Lemma 2.1: Each member of an $\mathcal{R}$ family expresses a set that can be used to provide the truth conditions for a sentence of an infinite liar-like set of sentences.
>
> Proof of Lemma 2.1: Since each member is recursive, the set of ordered pairs that is the graph of that function is recursive and so is the set of ordered pairs that results if, in that graph, we replace each natural number *n* in the first position of each ordered pair with "sentence with index *n*" and replace each 0 in the second position of each ordered pair with "is untrue" and replace each 1 in the second position of each ordered pair with "is true." The set resulting from these substitutions is exactly the kind suitable for giving the truth conditions for one of an infinite set of liar-like sentences. Since the member functions of an $\mathcal{R}$ family are recursive, they can be expressed by sentences in the language of arithmetic, and sentences of the language of arithmetic can be translated into English sentences.
>
> Claim 2.2: With each $\mathcal{R}$-$\mathcal{P}$ family there can be associated an infinite set of English sentences that forms a paradox.
>
> Proof of Claim 2.2: From lemma 2.1, we see that each member of an $\mathcal{R}$-$\mathcal{P}$ family can be translated into an English sentence that gives the truth conditions for a sentence of an infinite liar-like set. So if, for each *i*, we translate the i^{th} member function

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6 Intuitively, there must be some sort of restriction on the functions that are the members of a family because we want those functions to correspond to (and so be translatable into) English sentences. So the member functions of a family should at least be restricted to the class of recursively enumerable functions because, roughly, those functions are specifiable with a finite string of characters. So, such functions could be translated into a finite sequence of English words. On the other hand, it might be a bit more realistic to either take the range of each d_i to be {0,1,2} corresponding to *untrue*, *true* and *neither true nor untrue* (meaning that the i^{th} sentence makes no claim about the j^{th} sentence if $d_i(j) = 2$) or require that each d_i be only *recursively enumerable* rather than *recursive*. In the later case, the domain of each d_i would not be necessarily all of the natural numbers greater than i, we could say that d_i is defined at *j* iff i^{th} sentence makes a truth claims of the j^{th} sentence. Or finally, we could say that each d_i was only *r.e. and* that the range of the each d_i was {0,1,2}. I believe the non predicatively circular paradox I try to demonstrate in the sequel and the method I use to construct it would go through, with slight modifications, with these strengthened models. But more importantly for my argument, I consider a non predicatively circular paradox by trying to show that the number of $\mathcal{R}$-$\mathcal{P}$ families (a subset of each of the other families canvased in this footnote) is uncountable. It follows that the whole set of families considered here is uncountable and so a non predicatively circular paradox would exist among the extended set of families.

A related worry is that I have given only sufficient conditions for paradoxes of this sort. There are sets of sentences that do not correspond to an $\mathcal{R}$-$\mathcal{P}$ list but are in fact paradoxical. We will see an example in the conclusion. I do not think I have to address this issue in my argument because I'm trying to show that there is non predicatively circular paradox *even if we restrict our attention to a single kind of sentences* which might make up a list of sentences of a semantic, liar-like paradox, so of course it follows that there will be a non-circular paradox if we consider more the *exotic* kinds of sentences of which these paradoxes could be comprised.

into an English sentence that gives these truth conditions (which is the sentence with index *i*), we will have an infinite liar-like set of sentences. This set is paradoxical because there is no truth valuation for each of its sentences. To see this, note first that none of the sentences that correspond to semi-nullish member functions can be true without contradiction by Yabloesque reasoning. The only remaining sentences to consider are those which correspond to the triangulating member functions. None of these can be true without contradiction because if the sentence with (say) index *i*, were true, then for some *j* (> *i*) and *k* (> *j*) the sentences with indices *j* and *k* would both be true, but if the sentence with index *j* were true then the sentence with index *k* would be untrue, a contradiction. Each of the sentences of the set cannot be untrue, because, again by Yabloesque reasoning, if d_n is a nullish member, the sentence with index *n* would be true. There is no truth valuation for this set. ■

CHAPTER 3
THERE IS A PARADOXICAL SET OF SENTENCES THAT DOES NOT CONCERN A CIRCULAR PREDICATE

Priest claims that the "situation involved in Yablo's paradox, however formulated, is intrinsically circular, in exactly the same way that those involved in more familiar paradoxes of the family are." (Priest, 240) In this chapter, we take the final steps in showing that there is a liar-like paradox that is not predicatively circular. The paradox is unfamiliar and must remain so because it avoids Priest's intrinsic circularity charge by having a structure that is, roughly speaking, not recursive. This rough characterization will be made precise shortly; we will use the machinery we have set up in the previous section.

3.1 Technical Preliminaries and Presentation

Our plan is to show that there is a set of sentences that is paradoxical but not predicatively circular. I argued in the last chapter that any such set of sentences could equivalently be characterized as constructible from a recipe. If for *every* infinite set of sentences that were paradoxical, there were a method to construct the members of that set, and we help ourselves to *Church's Thesis* – the claim that for every algorithm or recipe there is a corresponding recursive function – we could claim of an arbitrary family, D_α, there is recursive function (f_α) such that f_α somehow encodes D_α. Since D_α is itself essentially a list of functions, one way that f_α could encode D_α is by encoding it is member functions: for each natural number n, $f_\alpha(n) = <d_{\alpha,n}>$ (where "$<d_{\alpha,n}>$" is natural number code, in an appropriate gödel coding, of the function $d_{\alpha,n}$). Given this encoding of D_α, we could characterize the entire set of paradoxical infinite sets of sentences with a set of recursive functions. To spell out more thoroughly the present suggestion, say that for any $\mathcal{R}$-$\mathcal{P}$ family, D_α, if $<d_{\alpha,i}>$ is a gödel code of $d_{\alpha,i}$ (the i^{th} member of D_α), then there is a recursive function, f_α, such that for each i, $f_\alpha(i) = <d_{\alpha,i}>$. In this scheme f_α can be said to be the recursive description of $\mathcal{R}$-$\mathcal{P}$ family D_α. And so there is a set, F, of recursive functions that includes an encoding function for each $\mathcal{R}$-$\mathcal{P}$ family. Since the set of recursive functions is countable[1], and F is a subset of this set, F is countable. Our assumption is that each member of F encodes an $\mathcal{R}$-$\mathcal{P}$ family, and there is one member of F for each $\mathcal{R}$-$\mathcal{P}$ family, so the set of $\mathcal{R}$-$\mathcal{P}$ families must be countable. So to show that there is a family that is $\mathcal{R}$-$\mathcal{P}$ that is not specified by a recursive function[2], it suffices to show that the set of lists that are $\mathcal{R}$-$\mathcal{P}$ is not countable. That's what we turn to next.

> Lemma 3.1: For any n, if $\{D_1, D_2, D_3, \ldots, D_n\}$ is a partial list of the $\mathcal{R}$-$\mathcal{P}$ families, then there is a partial list, $\{D_1, D_2, D_3, \ldots, D_n, D_{n+1}\}$, of the $\mathcal{R}$-$\mathcal{P}$ families such that the n+1st member of D_{n+1} is not nullish.
>
> Proof of Lemma 3.1: Consider a set ($\mathcal{D}$) of families each of whose members are nullish except the n+1st. Each of $\mathcal{D}$ is $\mathcal{R}$-$\mathcal{P}$ because the n+1st member must be either a triangulator or semi-nullish, *and* $\mathcal{D}$ is infinite (because there are infinitely many appropriate triangulators). For any n, if $D_1, D_2, \ldots, D_n$ is a partial list of the $\mathcal{R}$-$\mathcal{P}$ families, then we can form the partial list $\{D_1, D_2, \ldots, D_n, D_{n+1}\}$ by letting D_{n+1} be a member of $\mathcal{D}$. We are guaranteed that there is such a family because the first partial list is finite. ■
>
> Claim (3.1) The set of $\mathcal{R}$-$\mathcal{P}$ lists is uncountable.
>
> Proof of Claim (3.1): Assume otherwise for contradiction. Since the set of $\mathcal{R}$-$\mathcal{P}$ families is countable and because of lemma 3.1, we can assume that they can be listed $D_1, D_2, \ldots$, so that for each m, D_{2m} is a family such that it is $2m^{th}$ member is not nullish. We will demonstrate $D^* = \{d^*_1, d^*_2, d^*_3, \ldots\}$, an $\mathcal{R}$-$\mathcal{P}$ family that is

[1] Because every recursive function can be paired with the appropriate Turing Machine, each of which can be uniquely encoded by an natural number.

[2] I will call this sort a "non-recursive $\mathcal{R}$-$\mathcal{P}$ family".

not among the proposed list. For even n, let d^*_n be nullish, and for odd n, if $d_{n,n}$ (the n^{th} member function of the n^{th} family of our proposed countable set) is nullish, let $d^*_n(n + 1) = 1$ and for all $x > n + 1$ let $d^*_n(x) = 0$, otherwise let d^*_n be nullish. D^* is $\mathcal{R}$-$\mathcal{P}$ because there are an infinite number of nullish members, and every member is semi-nullish. D^* is not one of $D_1, D_2, \ldots$, because for all n, $d^*_n \neq d_{n,n}$. Contradiction, the set of $\mathcal{R}$-$\mathcal{P}$ family is not countable. ■

Corollary1: There are uncountably many infinite sets of English sentences that form liar-like paradoxes.

Corollary 2: There is an infinite set of English sentences that form a semantic, liar-like paradox that is not underwritten by a circular predicate.

Corollary 1 follows by reasoning similar to that of the proofs of Lemma 2.1 and Claim 2.2. Corollary 2 follows by corollary 1 and the reasoning presented just before Lemma 3.1.

To point towards a Yabloesque paradox that does not concern a circular predicate (because it is not recursively constructible), we can gesture toward what D^* might look like were a corresponding list of English sentences given in the manner of Claim 2.2. The paradox is an infinite list of indexed sentences $\{s_0, s_1, \ldots\}$ such that,

(if n is even) s_n is "For $k > n$, s_k is untrue."

(if m is odd) s_m is "For $k > m$, s_k is untrue." (OR) "s_{m+1} is true, and for $k > m + 1$, s_k is untrue."

3.2 Is The Foregoing Really *Not* Predicatively Circular?

With the hint of the set of sentences based on D^* in mind, we now argue that the class of Yabloesque sequences which are not recursive are not predicatively circular. Our strategy is the following. I will present an informal argument that there is no truth valuation for the set of sentences based on D^*, then I will try to mimic Priest's method to show that this set concerns a circular predicate. I hope we will see that it is not the case that there must be such a predicate.

Argument that sentence set based on non-recursive R-P list* D* *is paradoxical.

A. For arbitrary n, sentence with index n claims either that every sentence with index greater than n is untrue, or, that sentence with index $n + 1$ is true and each sentence with index greater than $n + 1$ is untrue.

B. For an even $p > n + 1$ sentence with index p claims that every sentence with index greater than p is untrue.

C. Sentence n cannot be true. If it were, then sentence p would be untrue, but every sentence with index greater than p would also be untrue. This exactly what sentence p claims, so sentence p would be true after all.

D. Index n was arbitrary so similar reasoning could be used to show that contradiction followed from the truth of any sentence of the set.

E. It cannot be that each of the sentences of the set is untrue, because in that case, since there is a sentence (let it is index be s) that claims that every sentence of higher index is untrue, and by hypothesis each of the these sentences would be untrue, sentence s would be true.

F. (Conclusion): the set is paradoxical.

First notice that in this argument (a similar argument could be run for any non-recursive sequence that met the $\mathcal{R}$-$\mathcal{P}$ condition), names of particular sentence never occur; only statements about the truth conditions of arbitrary sentences. Nevertheless, we can try, as Priest does, to "uncover" the logical form of the sentences. Instead of trying to formalize the argument with something similar to (1)-(11), let us note that were we to do that, we would have to do something like quantify over the "n" of string such as "s_n", and so we would reason, as Priest has, that this "n" must have been playing the role of variable. So I will skip straight to an attempt at carrying out Priest's method for discovering the circular predicate that is concerned by the set of sentences.

23. $\mathrm{SAT}(\mathsf{s}, n) \Rightarrow (k > n + 1)\neg\mathrm{SAT}(\mathsf{s}, k)$ (from (A))
24. for an even $p > n + 1$, $\mathrm{SAT}(\mathsf{s}, p) \Leftrightarrow (k > p)\neg\mathrm{SAT}(\mathsf{s}, k)$ (from (B))
25. $\mathrm{SAT}(\mathsf{s}, n) \Rightarrow \neg\mathrm{SAT}(\mathsf{s}, p)$ (from (23))

26. SAT(s, n) $\Rightarrow$ ($k > p$)$\neg$SAT(s, p) (from (23))
27. $\Rightarrow$ SAT(s, p) (from (24) and (26))
28. $\Rightarrow$ $\neg$SAT(s, n) (from (25) and (27))
29. (n)$\neg$SAT(s, n) (gen. based on (23)-(28))
30. ($k > p$)$\neg$SAT(s, k) (reasoning about "for all")
31. SAT(s, p) (from (24) and (30))
32. ✕ ((29) and (31))

Now the most we can say about this supposedly concerned predicate is that its satisfaction conditions can be represented as:

33. SAT(s, n) $\Leftrightarrow$ ($k > n$)$\neg$SAT(s, k) iff n is even
34. SAT(s, n) $\Rightarrow$ ($k > n + 1$)$\neg$SAT(s, k) iff n is odd

There *could* be a predicate here of whose satisfaction conditions we can assert (33) and (34), but we cannot give necessary and sufficient satisfaction conditions for it, because of the way that a non-recursive $\mathcal{R}$-$\mathcal{P}$ list was defined. So by Priest's lights, we cannot assert that the set concerns any specific predicate because, according to his presentation it seems that we must know the necessary and sufficient satisfaction conditions (or at least logical equivalents) to determine that a circular predicate is concerned by the set in question. In the present case, we do not know what the predicate is, and so our claim that whatever it is it must be concerned seems arbitrary.

Perhaps we should a different argument to the effect that there are *two* predicates (one corresponding to the nullish member functions and one corresponding to the semi-nullish member functions) one of which must be concerned by every member of the sentence set. A first try using two predicates might be the following.

35. SAT(s, n) iff ($k > n$)$\neg$SAT(s, k) and

36. $\text{SAT}(\mathsf{s}^*, n)$ iff $\text{SAT}(\mathsf{s}^*, n + 1)$ & $(k > n + 1)\neg\text{SAT}(\mathsf{s}^*, k)$

This approach won't work because the sentences were meant to quantify over *each* of those that followed. In the case of (35) and (36), it seems that sentences which concern s, for instance, do not refer to those which refer to s^*. Perhaps there is another work around – to make the predicate disjunctive in the following sort of way.

37. $\text{SAT}(\mathsf{s},n)$ iff $(k>n)\neg\text{SAT}(\mathsf{s},k)$ ***or*** $\text{SAT}(\mathsf{s},n+1)\&(k>n+1)\neg\text{SAT}(\mathsf{s},k)$

This approach does not work either because we need s to have the non-disjunctive satisfaction condition (for at least some natural numbers) "$\text{SAT}(\mathsf{s}, n)$ iff $(k > n)\neg\text{SAT}(\mathsf{s}, k)$" for step (24) of the reasoning presented in (23)-(32) to go through.[3] The possibility of a "closed form" presentation of the satisfaction conditions of predicate s is not available because the nature of the $\mathcal{R}$-$\mathcal{P}$ family on which this sentence set is based.[4] Perhaps, as a final attempt, we could claim that, in principle, there exist sentences that give the satisfaction conditions of s in the form of an infinite (non-recursive) list such as the following:

38. $\text{SAT}(\mathsf{s}, 0)$ iff $(k > 0)\neg\text{SAT}(\mathsf{s}, k)$

[3] Any hope for the the sort of predicate whose satisfaction conditions are given by (43) is lost because each family can include triangulating member functions. In any one $\mathcal{R}$-$\mathcal{P}$ family there can be infinitely triangulating member functions such that (if we think about functions set theoretically) no triangulator function is a subset of another. In that case, there is no hope for a closed form satisfaction condition for the predicate such a family might concerned.

[4] That is, we could not list such satisfaction conditions in a sort of closed form way because (first) there could be an $\mathcal{R}$-$\mathcal{P}$ list that contained an infinite number of different triangulators as in the previous footnote, and (second) D^* is non-recursive.

39. $\text{SAT}(\mathsf{s}, 1)$ iff $\text{SAT}(\mathsf{s}, 2)$ & $(k > 2)\neg\text{SAT}(\mathsf{s}, k)$
40. $\text{SAT}(\mathsf{s}, 2)$ iff $(k > 2)\neg\text{SAT}(\mathsf{s}, k)$
41. $\text{SAT}(\mathsf{s}, 3)$ iff $(k > 3)\neg\text{SAT}(\mathsf{s}, k)$
42. . . .

Though this sort of list might give satisfaction conditions of a predicate which could be mentioned by each sentence of the set based on *D**, it is important to recall that the construction of *D** showed that there were an *uncountably* many non-recursive Yabloesque sequences which were paradoxical. We can reason that each of these sequences form a paradox in a manner similar to (A)-(F). Any attempt to uncover the logical form of these sentences *à la* Priest must rest on the assumption that the sentences of this sequence have a specific form (i.e. include names of the sentences that follow). But that such sentences have this form is left underdetermined by the argument we have presented, as names for specific sentences do not appear in it. So the assertion that each sentence refers to a single predicate is only hypothesis – it cannot be shown that each of the sentences of a member of this class must make reference to a *single* predicate with circular satisfaction conditions. As a matter of fact, we can stipulate that there is a sequence based on a non-recursive $\mathcal{R}$-$\mathcal{P}$ family each sentence of which refers to a different predicate, as reasoning similar to that presented in (A)-(F) will go for such a sequence. So in the case of a sequence based on a arbitrary non-recursive $\mathcal{R}$-$\mathcal{P}$ family, we have, so far, no reason to claim that the set is predicatively circular. Indeed, we have evidence that there are such sequences that are *not* predicatively circular.

Finally we should consider whether the $\mathcal{R}$-$\mathcal{P}$ condition itself could be the thing concerned by a liar-like set of sentences which is based on an $\mathcal{R}$-$\mathcal{P}$ family. The answer seems to be no. A liar-like set of sentences could not concern a predicate (if any there be) that described generally the properties of an $\mathcal{R}$-$\mathcal{P}$ list if we were to be able both to reason to a contradiction and if the sentences were to make only truth claims of each other, because the truth conditions of sentences that concerned that sort of predicate would not be the truth or untruth of other sentences of the set, but would rather be the truth conditions of their successors. This is the case because the $\mathcal{R}$-$\mathcal{P}$ condition was meant to constrain the truth conditions of a given sentence relative to the truth conditions of other sentences rather than specifying the truth conditions of particular sentences outright. It is not obvious that a set of sentences that concerned the predicate formulation of the $\mathcal{R}$-$\mathcal{P}$ condition could constitute a liar paradox.[5]

3.3 A Paradox That Is Referentially Circular But Not Predicatively Circular

One concern of the dialectic is the general concern over whether circularity is necessary for paradox. I have tried to show that there are infinite liar-like paradoxes that are not referentially circular that are also not predicatively circular. One last question may be whether the two types of circularity are independent of each other. We could answer this question affirmatively if we could show that there is a paradoxical infinite set of sentences that is referentially circular and not predicatively circular. To show that notice that if we change the sentence s_2 of the paradox based on D^* from “For $k > 2$, s_k is untrue.” To “For $k > 2$, s_k is untrue, and s_1 is untrue.” Now the corresponding digraph has a cycle, and the set is referentially circular.

[5] Each sentence would not claim truth or untruth of it is companions, but would make claims about their truth conditions – sentences of this sort are not the appropriate kind for liar-like paradoxes.

Finally, we may also notice now that sets of sentences which are either referentially circular or predicatively circular, both or neither can fail to form paradoxes.[6] It seems that even though inferential circularity is required, paradox is independent from the referential and predicative circularity we have discussed in the preceding.

3.4 Conclusion

We tried to show in the third section of the first chapter that every liar-like paradox must be inferentially circular, that is, each sentence of must bear certain logical relationships to the other sentences. Because we have restricted our discussion to liar-like paradoxes, we know that these logical relations must be borne in virtue of a sentence's making truth claims about other sentences of the paradox. Sentences which bear the appropriate relations may be generated by a paradox with a referentially circular structure, which means that a sentence of the set refers indirectly to itself, by means of its truth claims and the attributions made by sentences of which it makes truth claims. Given that the "punch" of liar-like paradoxes is not a result of vagueness or other vagaries of natural language, there exists a graph theoretic tool to detect referential circularity and another to detect paradoxicality of referentially circular sets of sentences. I believe that so separating the search for indirect self-reference and the conditions under which a set of sentences forms a paradox is generally more informative than a proof theoretic method like Tennant's. Sets of sentences which make only truth claims about each other might be paradoxical without being referentially circular and *vice versa*.

[6] Examples of non paradoxical sets of sentences that are (1) predicative, non-referential: a set like Yablo's except substitute "true" for "untrue"; (2) referential, non-predicative: a set of sentences based on a non-recursive $\mathcal{R}$ family that contains only one semi-nullish member function (which is also nullish) and that is slightly modified such that its graph has a cycle; (3) both: a set exactly like the predicative, non-referential set above except that one sentence is modified such that the graph has a cycle; (4) neither: a set of sentences based on the non-recursive $\mathcal{R}$ list above in (2) without the modification that caused the cycle in the graph.

It turns out on the present suggestion that *YP* is not referentially circular, but is, as Priest argues predicatively circular. The process of assessing the methods of Priest's argument that *YP* is underwritten by a circular predicate helps us see that if a set of sentences is predicatively circular, that set of sentences must be constructible from a recipe. Since we can imagine what's minimally required for a set of sentences to lack an assignment of truth-values, we can formalize with families of recursive functions the behavior of a subset of infinite liar-like paradoxes. Once we have seen how this might come off, it is a short step to showing that there must be a non-recursive list of functions which meets the formal requirements for paradox because such families form an uncountable set. One result of the whole adventure is that we can provide the *sketch* of a paradox the sentences of which is not predicatively circular. And so it does not seem that predicative circularity is a necessary feature of liar-like paradoxes.

It might be thought we could go further and offer an analysis of paradoxicality in the context of infinite sets of the sentences with just a bit more development of the notion of $\mathcal{R}$-$\mathcal{P}$ families. But I think we will see that it would take significantly more work to come up with necessary and sufficient conditions for such paradoxicality. To see this, consider an infinite list of sentences indicated by the following schema:

43. s_n: $\neg(\exists x > n)(\forall y > x)\ s_y$ is true.

Another rendition of this sentence might be "s_n: 'infinitely many sentences of index greater than n are untrue.'" This list of sentences forms a paradox: if s_0 is true then for every $m > 0$ s_m is true because there are always an infinite number of sentences with index greater than m which are untrue, but in this case s_0 must be untrue, there being no untrue sentences that follow it. On the other hand if s_0 is untrue then there are only finitely many sentences with higher index that are untrue, but this situation leads to contradiction because there must be some $p > 0$ such that s_p is true, but this is the case only if infinitely many of the sentences that follow s_p are untrue, in which case s_0 would have been true after all. Similar reasoning holds for sentences with arbitrary index.

This set of sentences is paradoxical but is not captured by the $\mathcal{R}$-$\mathcal{P}$ family characterization. What has gone wrong? Even though this set of sentences is predicatively circular, the sentences do not recursively describe a *single* set of truth-values for the subsequent sentences (as did each member of an $\mathcal{R}$-$\mathcal{P}$ family formally as a function from a subset of the natural numbers to $\{0,1\}$), rather the sentences describe *sets* of truth-values for all subsequent sentences. To illustrate, (43) describes sets of truth-values for all subsequent sentences in which there are infinitely many untrue sentences. Part of the "bite" comes from the fact that for any natural numbers n and m, the intersection of the sets described by n^{th} and m^{th} sentences of such a set is non-empty and so if $n = 0$, and the first sentence is true then for any natural number the m^{th} must be true. All the subsequent sentences must be true, but each claims that infinitely many of the following are untrue. A general characterization of *this sort* of phenomenon *might* be available – something, for example, that made more explicit use of set theory – but that project exceeds the scope of what we would set out to do here.

Finally, a few last points about the possibility of formal results available from paradoxes comprising infinite sets of liar-like sentences. If the demonstrations offered in the appendices are successful, the result is that using the notion of $\mathcal{R}$-$\mathcal{P}$ families we can prove both an undefinability theorem similar to Theorem 1 of Mostowski, Robinson and

Tarski (1953)[7] and an incompleteness theorem similar to that given by Smullyan (1992).[8] Theorem 1 of the former work shows that if formal system ***T*** is to be consistent then the set ***V*** of all and only those numbers which are gödel codes of valid sentences in ***T*** and diagonal function ***D***[9] cannot both be definable. *Appendix B* shows that there is a set ***Y*** defined with the help of predicate that simulates an $\mathcal{R}$-$\mathcal{P}$ family that cannot be definable along with ***V*** if the formal system in which they're defined is to be consistent. *Appendix C* shows that for certain systems in which an $\mathcal{R}$-$\mathcal{P}$ list can be defined, there is a gödel numbering for the expressions of that language relative to which there are infinitely many undecidable sentences for that system. It is interesting to note that both of these results to nothing to lower the standard that's required for the existence of an undefinable set within a system or an undecidable sentence. In each case, undefinable sets and undecidable sentences would already be features of systems which were powerful enough to define $\mathcal{R}$-$\mathcal{P}$ families, i.e. systems that were capable of defining recursive sets. Another point of interest is that the non-recursive $\mathcal{R}$-$\mathcal{P}$ families we labored so long to demonstrate cannot be used in either formal result. Both appendices require that the $\mathcal{R}$-$\mathcal{P}$ family in question be recursive, otherwise in the case of *Appendix B* we could not speak of "$<\psi>$", and in the case of *Appendix C* we could not recover the sequence of gödel codes of s_0, s_1, s_2, ... So it is unclear whether any additional technical result is available from the existence of non-recursive $\mathcal{R}$-$\mathcal{P}$ families

[7] For all references to Mostowski, Robinson and Tarski see (Mostowski, Robinson and Tarski 1953)

[8] For all references to Smullyan see (Smullyan 1992).

[9] "*Dn* is the number correlated with the expression which is obtained from E_n by replacing everywhere the variable *u* by the term Δ_n." (46) For example, if, for one place predicate *H*, $<H(x)> = n$, then $Dn = <H(n)>$.

APPENDIX A
USING DIGRAPHS TO DETERMINE WHEN A SET OF SENTENCES FORMS A PARADOX

We give a method for determining a sufficient condition for whether a set of referentially circular sentences forms a paradox using ***A***, an algorithm which colors in stages (nondeterministically) the vertices of ***G*** with color_1 and color_2.

A {

Let *x*, *y* be a variables ranging over the natural numbers. Let *T* be a variable ranging over sets. Set $T \leftarrow \varnothing$. Choose *i* to be the least index of the vertices of ***G***. Set $x \leftarrow i$. Assign $\boldsymbol{V_x}$ either color_1 or color_2.

While there is some vertex $\boldsymbol{V_j}$ such that $j \notin T$ {

If $\boldsymbol{V_x}$ is color_1, consider which assignment of truth values to all and only the sentences whose corresponding vertices share edges *from* $\boldsymbol{V_x}$ will make sentence ***i*** true (these sentences must be all and only those sentences to which sentence ***i*** directly refers by construction of ***G***), color (nondeterministically) vertices of this set with color_1 whose corresponding sentences must be true in order for sentence ***i*** to be true, color vertices of this set with color_2 whose corresponding sentences must be untrue in order for sentence ***i*** to be true. All the vertices that share edges from $\boldsymbol{V_x}$ may not be colored.

If $\boldsymbol{V_x}$ is color_2, consider which assignment of truth values to all and only the sentences whose corresponding vertices share edges *from* $\boldsymbol{V_x}$ will make sentence *i* untrue, color (nondeterministically) the vertices of this set with color_1 whose corresponding sentences must be true in order for sentence *i* to be untrue, color the vertices of this set with color_2 whose corresponding sentences must be untrue in order for sentence *i* to be untrue. All the vertices that share edges from $\boldsymbol{V_x}$ may not be colored.

Choose one of the vertices, say $\boldsymbol{V_y}$, just colored. If $x \neq y$, set $T \leftarrow T \cup \{x\}$ (the color of $\boldsymbol{V_x}$ is "fixed" iff $x \in T$). Set $x \leftarrow y$. }

}

Claim A.1: A graphically circular set of sentences is paradoxical if the corresponding graph has no "stable coloring" in the following sense. For any execution of $\boldsymbol{A}$ on the corresponding previously uncolored graph $\boldsymbol{G}$, *either* there are colored vertices $\boldsymbol{V_i}$ and $\boldsymbol{V_j}$ such that $i \in T$ but $j \notin T$ (that is $\boldsymbol{V_i}$ has been assigned $color_1$ ($color_2$) at some previous stage and $\boldsymbol{V_j}$ has *just* been assigned a color and that color *hasn't* been fixed), there is an edge from $\boldsymbol{V_j}$ to $\boldsymbol{V_i}$ and $\boldsymbol{A}$ assigns $\boldsymbol{V_i}$ $color_2$ ($color_1$) *or* the color of a vertex which cannot be fixed alternates endlessly on the execution of $\boldsymbol{A}$.[1]

Proof of Claim A.1: Assume otherwise for contradiction: that there is a set, $\boldsymbol{R}$, of sentences which is not paradoxical but whose corresponding graph $\boldsymbol{G}$ contains a cycle, yet has no stable coloring. If a set of sentences each of which only makes truth claims of the others is not paradoxical then there is an assignment of "true" and "false" to those sentences that does not lead to contradiction. For each x, if vertex V_x is a vertex of G, and if V_x were colored with $color_1$ just in case the x^{th} sentence of R were true and with $color_2$ just in case the x^{th} sentence of R were untrue, then this coloring would be stable in terms of $\boldsymbol{A}$, in contradiction to the original assumption. ■

1 A vertex cannot be fixed in case the corresponding sentence is directly self-referential.

APPENDIX B
AN UNDEFINABILITY THEOREM

In our development, we're far from that which is usually considered a semantic paradox. I want to suggest that what we're doing is not so strange in light of how traditional liar sentences can be used in the study of formal systems. Mostowski, Robinson and Tarski's (Mostowski, Robinson and Tarski 1953) Theorem 1 shows that any formalized theory (call it ***T***) in which both ***V*** (the predicate applying to codes of valid sentences) and function ***D*** (a diagonal function) are definable must be inconsistent. We will demonstrate here that there is a formula with one free variable which defines a set of natural numbers ***Y*** such that ***V*** and ***Y*** are not both definable if the theory ***T*** of the formal system in which they're to be defined is consistent. Let ψ_α be a formula of the language of theory ***T*** with one free variable. Now, let ***Y*** be a predicate of natural numbers such that $<\psi_\alpha(m)> \in Y$ just in case for all $p > m$, $<\psi_\alpha(p)> \in Y$ if $d_{\alpha,m}(p) = 1$ and $<\psi_\alpha(p)> \notin Y$ if $d_{\alpha,m}(p) = 0$.

<u>Claim A.2</u>: If ***T*** is consistent, then ***Y*** and ***V*** are not both definable.

<u>Proof of Claim A.2</u>: Suppose for contradiction that there is some predicate ***Y*** so defined by ψ_α and predicate ***V*** such that $n \in \boldsymbol{V}$ iff the sentence of ***T*** whose gödel code is n is valid. If $<\psi_\alpha(0)> \in \boldsymbol{Y}$ is valid, we see that it must be the case that, by construction of D_α, that for some $m > 0$, we can infer that $<\psi_\alpha(m)> \in \boldsymbol{Y}$ and $<\psi_\alpha(m)> \notin \boldsymbol{Y}$, a clear contradiction. If, on the other hand, none of $\boldsymbol{Y}(0)$, $\boldsymbol{Y}(1)$, $\boldsymbol{Y}(2)$, . . . are valid, then since there is some m such that $d_{\alpha,m}$ is nullish and $<(\forall x)(\neg \boldsymbol{Y}(x))>$ is in ***V***, we have warrant to infer $<\psi_\alpha(m)> \in \boldsymbol{Y}$, but in this case, $<\boldsymbol{Y}(<\psi_\alpha(m)>)> \in \boldsymbol{V}$, and so we can infer $<(\forall x)(\neg \boldsymbol{Y}(x))>$ is *not* in ***V***, in contradicition to our assumption. ***Y*** and ***V*** cannot both be definable. ■

APPENDIX C
AN SMULLYAN STYLE INCOMPLETENESS THEOREM

Using an $\mathcal{R}$-$\mathcal{P}$ family D_α, we can generate an incompleteness result. Let L be a language similar to the one developed by Smullyan (1992, chp. 1-4). Following Smullyan, if there is an arithmetically definable proof procedure based on some axiom schemas and rule of inference, then we can speak of a one place predicate $\boldsymbol{P}x$ which defines proofs is satisfied by all and only the gödel numbers (for any suitable numbering) of the sentences of L which are provable according to this procedure. If the negation of sentence is provable then the negatum is *refutable*. If the system determined by the language and the proof procedure is consistent, then no sentence is provable and refutable. If $\boldsymbol{P}x$ satisfies the following two additional conditions then we can get a kind of incompleteness result: (1) if $\boldsymbol{P}(<\boldsymbol{P}(x)>)$, then $\boldsymbol{P}(x)$, and (2) for metalinguistic variables α, β and δ, if $\boldsymbol{P}(<\delta>)$ and $\delta = \ulcorner(\alpha \;\&\; \beta)\urcorner$ then $\boldsymbol{P}(<\alpha>)$ *and* $\boldsymbol{P}(<\beta>)$, and if $\boldsymbol{P}(<\delta>)$ and $\delta = \ulcorner\neg\alpha\urcorner$, then $\neg\boldsymbol{P}(<\alpha>)$. First, We can draw attention to an infinite set of sentences (denoted metalinguistically for convenience as) $s_0, s_1, s_2, \ldots$ of L which have the following distinction. For each j, and a recursive $\mathcal{R}$-$\mathcal{P}$ family D_α,[1] let s_j be the shortest sentence of L containing the symbols "$\boldsymbol{P}$", numerals for odd numbers greater than j, the symbols required to describe recursive functions and the usual first order logic connectives such that it is logically equivalent to $\ulcorner(s_j \;\&\; \boldsymbol{P}(2k + 1)\urcorner$ iff $d_{\alpha,j}(k) = 1$ $(k \geq j)$ and is logically

[1] If the family is not recursive then there is no way to recover the numbers which are gödel codes of the sentences that "define" the behavior of P on the odd numbers. If we're to appropriately modify the godel coding in an effective fashion, then we must be able to recover that sequence.

equivalent to $\ulcorner(s_j \mathbin{\&} \neg \boldsymbol{P}(2k+1)\urcorner$ iff $d_{\alpha,j}(k) = 0$. Essentially, each s_j recursively describes the behavior of $\boldsymbol{P}$ for each of the odd numbers greater than $2j$ in a manner similar to that in which $d_{\alpha,j}$, if considered as outlined in Lemma 2.1 and Claim 2.2, indicates which of the subsequent sentences are true or untrue. For each j, s_j is unique because it is the shortest such sentence of it is logical equivalence class. For example, if $d_{\alpha,j}$ is nullish then s_j is the shortest sentence of L that is logically equivalent to "$(\forall x)((\forall y)(((x > 2j) \mathbin{\&} (x = 2y-1)) \rightarrow \neg \boldsymbol{P}(x))$". Now let g be a recursive, 1–1, onto gödel numbering function from the expressions in the vocabulary of L to the *even natural numbers*. We will construct a new gödel numbering function, g', that's suited to our purposes. Let g', be the same as g except let $g'(s_0) = 1$, $g'(s_1) = 3$, and in general $g'(s_n) = 2n + 1$. Now g' is gödel numbering function from the expressions of L to a subset of the natural numbers whose compliment is an infinite subset of the even natural numbers.[2]

<u>Claim A.3</u>: If the system based on L and $\boldsymbol{P'}$ is ω–consistent, then there is a sentence, s, of L such that neither $\boldsymbol{P'}(<s>)$ nor $\boldsymbol{P'}(<\ulcorner\neg s\urcorner>)$.

<u>Proof of Claim A.3</u>: Consider s_i where $d_{\alpha,i}$ is nullish. If $\boldsymbol{P'}(<s_i>)$ it follows that for a nullish $d_{\alpha,j}$ $(j > i)$, $\boldsymbol{P'}(<\neg\boldsymbol{P'}(<s_j>)>)$ and so $\neg\boldsymbol{P'}(<s_j>)$. If $\boldsymbol{P'}(<\ulcorner\neg s_j\urcorner>)$, then we would have $\boldsymbol{P'}(<(\exists x)(\ \exists y)((x > 2j) \mathbin{\&} (x = 2y-1) \mathbin{\&} (\boldsymbol{P'}(x))>)$ and since our system is ω–consistent, we have $\boldsymbol{P'}(<\boldsymbol{P'}(k)>)$, and so $\boldsymbol{P'}(k)$ for an odd k greater than $2j$. But we have already seen that from $\boldsymbol{P'}(<s_i>)$ it follows that for this k, $\boldsymbol{P'}(<\neg\boldsymbol{P'}(<k>)>)$, i.e. that $\neg\boldsymbol{P'}(k)$, a contradiciton, so if $\boldsymbol{P'}(<s_i>)$ then neither $\boldsymbol{P'}(<\ulcorner\neg s_j\urcorner>)$ nor $\boldsymbol{P'}(<s_j>)$. Now if $\boldsymbol{P'}(<\ulcorner\neg s_i\urcorner>)$ then we would have, because of ω–consistency, $\boldsymbol{P'}(<s_m>)$ for some $m > i$. Now if $d_{\alpha,m}$ is semi-nullish, then we see that by reasoning similar to the forgoing there is an undecidable sentence whose code number is odd and greater than m. On the other hand, if $d_{\alpha,m}$ is not semi-nullish, a contradiction results

2 By using this coding function we lose the "compositional" feature of, for example, Boolos and Jeffrey's system of coding. Specifically, if s_0 were "$(\forall x)((\forall y)(((x > 2) \mathbin{\&} (x = 2y-1)) \rightarrow \neg P'(x))$", then in a normal coding system, the gödel number of s_0 would be determined from the gödel numbers of each of "P'", "(", "&" etc. in a compositional fashion – for instance by simply concatenating of the gödel numbers in of each of the constituent symbols of "$(\forall x)((\forall y)(((x > 2j) \mathbin{\&} (x = 2y-1)) \rightarrow \neg P'(x))$". To get $\boldsymbol{P'}$ to be about the right sentences, we cannot maintain this compositionality, but there is no problem because the inverse of g' is well defined.

because there are n, $p > m$ such that $d_{\alpha,m}(n) = 1$ and (WLOG) $d_{\alpha,m}(p) = 1$ and $d_{\alpha,n}(p) = 0$. In this case, from $\boldsymbol{P}'(<s_m>)$ it follows that $\boldsymbol{P}'(<s_n>)$ *and* that $\boldsymbol{P}'(<s_p>)$, but from $\boldsymbol{P}'(<s_n>)$ it follows that $\neg\boldsymbol{P}'(<s_p>)$. ■

Corollary A.3: For this system and gödel numbering, there are infinitely many undecidable sentences which are not logically equivalent.

Proof of Corollary A.3: For a recursive $\mathcal{R}$-$\mathcal{P}$ family there are infinitely many nullish member functions, Claim A.3 shows that for each such function there is a sentence that is undecidable relative to the system and corresponding gödel numbering in question and, as represented in this system, no two distinct nullish member functions are logically equivalent. ■

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BIOGRAPHICAL SKETCH

I graduated in 1995 from Brown University with an A.B. in mathematics and in 2001 from The University of Tennessee, Knoxville with an M.S. in computer science.