Paradox without satisfaction

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1. *Introduction*

Consider the following denumerably infinite sequence of sentences:

(s_1) For all $k > 1$, s_k is not true.
(s_2) For all $k > 2$, s_k is not true.
(s_3) For all $k > 3$, s_k is not true.
. . .
. . .
. . .
(s_n) For all $k > n$, s_k is not true.
. . .
. . .
. . .

According to Stephen Yablo, the above list generates a liar-like paradox *without* circularity (see Yablo 1985 and 1993). After all, in contrast with the usual liar paradox, no sentence in the Yablo list refers to itself, and as opposed to well-known liar cycles,[1] no sentence refers to sentences above it in the list. However, similarly to the (usual or cyclic) liar, a contradiction is derivable from the list.

But is Yablo's paradox really non-circular? To this question, Graham Priest gave a surprising answer. In his view, despite initial appearances, the paradox *is* circular (Priest 1997). After all, if the formulation of the paradox doesn't seem to involve circularity, the *argument to contradiction* definitely does. As Priest argued, Yablo's paradox has 'a fixed point … of exactly the same self-referential kind as in the liar paradox'. As a result, 'the circularity is … manifest' (Priest 1997: 238).[2]

In this paper, we challenge Priest's answer – in a new way. To the best of our knowledge, everyone in the debate has conceded the adequacy of Priest's reconstruction of the argument to contradiction. We point out a limitation that hasn't been noted. Priest's argument requires the existence of a satisfaction relation that plays the role of a fixed point in Yablo's

[1] Here is a typical example of such cycles:

(1) Sentence (2) is not true.
(2) Sentence (1) is true.

[2] Not surprisingly, Priest's argument generated a lively debate. The main papers are Sorensen 1998, which defends the non-circularity of Yablo's paradox, and Beall 2001, which sides with Priest against Yablo and Sorensen.

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paradox. However, if a contradiction can be established from the Yablo list *without* invoking such a relation, there's no fixed point, and so Priest's argument is blocked. As we show below, as opposed to Priest's claim, the argument to contradiction *doesn't* require the satisfaction relation; in fact, the argument goes through perfectly well without the latter. We conclude the paper by mentioning a consequence of this way of presenting the argument to contradiction for the significance of Yablo's paradox.

Before proceeding, it's important to note that so far the only argument for the circularity of the Yablo list is Priest's. But, as hinted above, and as will become clear in a moment, Priest obtains the fixed point from the *argument to contradiction*, *not* from the *construction of the list*. So, as far as Priest's argument is concerned, what we have to say here stands even if the fixed point turns out to be a feature of the list. After all, as we argue, at least the argument to contradiction *doesn't* require a fixed point. Now, whether the fixed point is a feature of the list or not is hard to tell, given that, to the best of our knowledge, no argument has been presented to support this claim. Our focus here is on Priest's challenge – the only challenge so far to the non-circularity of Yablo's paradox.

2. *Priest's argument*

First, let's review Priest's argument to contradiction. The sentences in the Yablo list can be formalized with a truth predicate, T, in the following way: for all natural numbers n, s_n is the sentence. $\forall k > n, \neg Ts_k$. Priest's argument to contradiction goes as follows (Priest 1997: 237): For every n,

$$\begin{aligned} Ts_n &\Rightarrow \forall k > n,\ \neg Ts_k \qquad (^*) \\ &\Rightarrow \neg Ts_{n+1}. \end{aligned}$$

But,

$$\begin{aligned} Ts_n &\Rightarrow \forall k > n, \neg Ts_k \qquad (^*) \\ &\Rightarrow \forall k > n+1, \neg Ts_k \\ &\Rightarrow Ts_{n+1}. \qquad (^{**}) \end{aligned}$$

Thus, given that 'Ts_n' entails a contradiction, we conclude that $\neg Ts_n$. Priest then notes:

> But n was arbitrary. Hence $\forall k\ \neg Ts_k$, by Universal Generalization. In particular, then, $\forall k > 0, \neg Ts_k$, i.e., s_0, and so Ts_0. Contradiction (since we have already established $\neg Ts_0$). (Priest 1997: 237)[3]

Having established the contradiction, Priest invites us to focus on the lines marked $(^*)$, and asks about their justification. He claims:

[3] Note the use of universal generalization *and* instantiation in this passage. We will return to this point later.

> It is natural to suppose that this is the T-schema, but it is not. The n involved in each step of the reduction argument is a free variable, since we apply universal generalization to it a little later; and the T-schema applies only to sentences, not to things with free variables in. It is nonsense to say, for example, T'x is white' iff x is white. What is necessary is, of course, the generalization of the T-schema to formulas containing free variables. ... This involves the notion of satisfaction. For the lines marked (*) to work, they should therefore read:
>
> $$S(n, s') \Rightarrow \forall k > n, \neg Ts_k$$
>
> where S is the two-place satisfaction relation between numbers and predicates, and s' is the predicate $\forall k > x, \neg Ts_k$ (Priest 1997: 237)[4]

A similar point also applies to the line marked (**). In this case, what we need is:

$$\forall k > n+1, \neg Ts_k \Rightarrow S(n+1, s').$$

And by rewriting every other line of the argument accordingly, replacing truth by satisfaction, the final contradiction – $\forall k > 0, \neg S(k, s')$ and its negation – is obtained.

It's now clear, Priest concludes, that

> the paradox concerns a predicate, s', of the form $\forall k > x, \neg S(k, s')$; and the fact that $s' =$ '$\forall k > x, \neg S(k, s')$' shows that we have a fixed point, s', here, of exactly the same self-referential kind as in the liar paradox. In a nutshell, s' is the predicate 'no natural number greater than x satisfies this predicate'. The circularity is now manifest. (Priest 1997: 238)

Even if we grant that the circularity is now manifest, the question still arises: is the circularity inherent in Yablo's paradox, or is it simply an artefact of the particular version of the argument to contradiction used by Priest? We think that the latter is the case.

3. *The argument reformulated*

We now show how to derive a contradiction from the Yablo list, without applying the T-schema to any open formula. And so there is no need to invoke a satisfaction relation to run Yablo's paradox. As will become clear, our argument is parasitic on Priest's argument, but with a crucial new twist.

Consider 's_1' in the Yablo list. Suppose 's_1' is true (which, as above, we denote by 'Ts_1').

$$\begin{aligned} Ts_1 &\Rightarrow \forall k > 1, \neg Ts_k \\ &\Rightarrow \neg Ts_2. \end{aligned}$$

[4] We have made minor changes to Priest's notation, but nothings hangs on this.

But,

$$
\begin{aligned}
Ts_1 &\Rightarrow \forall k > 1, \neg Ts_k \\
&\Rightarrow \forall k > 2, \neg Ts_k \\
&\Rightarrow Ts_2 .
\end{aligned}
$$

So, given that 'Ts_1' entails a contradiction, $\neg Ts_1$. This means that there is at least one true sentence in the Yablo list. Let the first such sentence be 's_i'. (Note that 'i' is not a variable, but an unknown, particular natural number.) Now consider 's_i'.

$$
\begin{aligned}
Ts_i &\Rightarrow \forall k > i, \neg Ts_k \\
&\Rightarrow \neg Ts_{i+1} .
\end{aligned}
$$

But,

$$
\begin{aligned}
Ts_i &\Rightarrow \forall k > i, \neg Ts_k \\
&\Rightarrow \forall k > i+1, \neg Ts_k \\
&\Rightarrow Ts_{i+1} .
\end{aligned}
$$

Thus, a contradiction can be derived from the truth or untruth of a particular sentence, 's_1', in the Yablo list.

Three remarks are in order here:

(a) The sentence we used to establish the contradiction is, of course, arbitrary – we could have started with any sentence in the list. But – and this is crucial – the arbitrariness of the sentence we started with is *never* used to derive the contradiction, and so it's *not necessary* for the argument to go through. Even if 's_1' were the only paradoxical sentence in the Yablo list, this would be sufficient to conclude that Yablo's paradox (i) is a paradox, and (ii) is not circular – or, at least, it's not circular in the sense at issue here (i.e. it doesn't require a fixed point of a self-referential kind).

(b) Priest's argument to contradiction is unnecessarily strong. The argument actually establishes that *every* sentence in the Yablo list is paradoxical. As we saw, to get this conclusion, Priest needs to use universal generalization at a crucial point in the argument, just to instantiate the result in the following line (see our first full quotation from Priest's paper in §2). By focusing on a *particular sentence* in the Yablo list, and then only establishing that there is *at least one* paradoxical sentence in the list, our argument bypasses this move altogether. After all, we don't need to prove that Yablo's paradox is massively paradoxical to establish that it is a paradox!

(c) Finally, in our argument, we have not used the T-schema on anything other than *sentences* in order to derive the contradiction. In particular, we have not illegitimately used the T-schema on any open formulae. As we saw, the contradiction can be derived from the Yablo list *without* invoking a satisfaction relation, and so the fixed point is only an artefact of the argument to contradiction used by Priest. To prove that there is at least one paradoxical sentence in the Yablo list, there's no need for a fixed point – this is the result established above – and without the fixed point, there's no reason to believe that there is circularity.[5]

4. Conclusion

What is the consequence of blocking Priest's Yablo's paradox? The most important one is that now there's no reason to believe that Yablo's paradox is circular. To the best of our knowledge, no circularity is presupposed either in the formulation of the paradox or in the argument to contradiction. And so, more generally, a case can be made that circularity and self-reference are *not* the only 'causes' of the semantic paradoxes. It's possible to have liar-like paradoxes without either circularity or self-reference. As a result, the main lesson from Yablo's paradox is that more work needs to be done in the analysis of the semantic paradoxes to diagnose exactly why they emerge.[6]

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References

Beall, JC. 2001. Is Yablo's paradox non-circular? *Analysis* 61: 176–87.
Priest, G. 1997. Yablo's paradox. *Analysis* 57: 236–42.
Sorensen, R. 1998. Yablo's paradox and kindred infinite liars. *Mind* 107: 137–55.
Yablo, S. 1985. Truth and reflection. *Journal of Philosophical Logic* 14: 297–349.
Yablo, S. 1993. Paradox without self-reference. *Analysis* 53: 251–52.

[5] According to Graham Priest (private communication), the crucial issue raised in his paper is how one knows that the Yablo list exists. In addition, as he argued, to establish the list's existence a fixed-point construction is required. However, it's not clear to us that the argument Priest gave establishes how we know that the Yablo list exists. Priest's argument seems to *presuppose* the existence of the list, in order to establish that to derive a contradiction from the latter, a fixed-point construction is required. What we have argued here is that a contradiction can be derived without the fixed point. How do we know that? By following the derivation presented above.

[6] Our thanks go to JC Beall and Graham Priest for extremely helpful discussions.