The New Clothes of Paradox

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Abstract

Any recursive definition without a base case is circular. However, Yablo's sequence, defined by a reverse recursive definition, has no base case. As it has no base case, it is circular. It is not a semantic paradox, and hence, can be formulated at an object language level. This study will demonstrate how to transform a valid recursive definition to an invalid recursive definition using definition examples step-by-step. In a nutshell, Yablo's paradox is generated by a vicious definition that is:

1. circular, because it is a reverse recursive sequence definition, without a base case;
2. entails a contradiction.

1. Introduction

At a first glance, the most significant element in the debates on Yablo's paradox has been the problem of circularity. However, on closer investigation, a second level of controversy emerges – that is, the selection of the formally correct language in which the sequence can be formulated, and following deductive inference, the contradiction derived. In fact, it would be a mistake to ignore this second level, that is, the existence of Yablo's endless sequence. What kind of entities are we talking about – a list of definitions or definiendums – that is, a sequence of sentences in which sentences are the truth-bearers?[1]

The approach adopted in this paper for the formulation of Yablo's list is, to some extent, similar to Thomas Forster's solution. However, he claimed: "The first thing to notice is that the proof of the paradox is infinitely long" (Forster, 1996). On the contrary, this study will demonstrate that there is a finite proof of the paradox in object language, by applying universal quantification instead of a denumerable

[1] This is not true for every sentence, but only for interpreted declarative sentences that have unambiguous information contents, such as "The Battle of Waterloo was fought on Sunday 18 June 1815 near Waterloo in present-day Belgium, then part of the United Kingdom of the Netherlands."

sequence of sentence operators. As far as the question of circularity is concerned, it is ambiguous; therefore, the answer cannot be definite either, as James Hardy pointed out in his discerning paper (Hardy, 1995). If the question of circularity refers to the truth conditions of the list of formulae in the first-order logic language, then the answer is definitely “No.” This study will demonstrate that Yablo’s paradox is not a semantic paradox. However, if the question of circularity refers to the definition that generates the Yablo’s sequence, then the fact that the definition is circular can be revealed. Regarding the first claim, consider any large, but finite initial segment of Yablo’s sequence. This list can be perfectly simulated in the spreadsheet computer software: the software will alert us if it is asked to execute a circular calculation. As the spreadsheet model clearly shows, the truth conditions of the finite list would be circular only if the list of natural numbers could be circular in contrast to the Peano axioms.[2] When the problem is simulated using the spreadsheet software, it is visible on the screen that the finite list is not circular, because we do not get an alert message from the computer. Laurence Goldstein (2006) and Laureano Luna (2009) pointed out that the reverse recursive definition of Yablo’s sequence is not well-founded (it has no base case). Graham Priest was the first to argue that Yablo’s paradox is circular, and his argument proceeds in another direction and is connected to a fixed point construction, which will not be analyzed here. The third question also comes from Priest: “How can one be sure that there is such a sequence?” (Priest, 1997). This study will prove that Yablo’s sequence of definiendums does not exist if we restrict the domain of the sequence to natural numbers, although this fact should not mislead one into thinking that a sequence of definitions does not exist. Discussing the existence of Yablo’s sequence, J.C. Beall wrote: “Nobody, I should think, has seen a denumerable paradoxical sequence of sentences …” (Beall 2001). Fortunately, that power of imagination is not necessary: a little less is sufficient. Imagine, “… At the gates of Heaven an infinite queue of people is tailed back” (Priest 1997), tirelessly putting on hats and taking them off again – we will understand why later.

[2] The spreadsheet model is downloadable from: http://www.andrasek.hu/ferenc/models/pwsr-fin.xls

2. Explication

To make matters clearer, let us start the enquiry by elucidating some problems regarding definitions and then proceed to an investigation of the truth values of three second-order logic formulas.

2.1. A few notes on definitions

In his original paper, Stephen Yablo's intention was to construct an explicit stipulative definition of a sentence sequence.[3] There is good reason to investigate this kind of definition focusing on possible flaws. In the present paper, we investigate the definitions, which, on the left-hand side (i.e., the definiendum), have a simple, unary predicate or function. This sort of a definition may suffer from many diseases, such as:

i. Emptiness

The right-hand side of the definition might be a contradictory formula; thus, the extension of the definiens is empty and cannot be applied to anything. This is not harmful until one supposes that something satisfies the definiens. This kind of definition is superfluous, but does not infect one's theory. For example, $F(x) \leftrightarrow_{\mathrm{df}} x$ is a natural number and $3 \neq x$ and $3 = x$.

This indicates that some x natural number is F, if and only if x is equal and not equal to 3. As there is no such number, the predicate "F" does not refer to anything.

ii. Circulus vitiosus

Only the same free variables can play on the left and right side of the definitions, and apart from recursive definitions, the left-side functor – predicate or function – is not allowed to occur on the right side. In other words, it is not permissible for the definiens to comprise the definiendum.[4] We consider "circular definition" as the violation of this rule. For example, $F(x) \leftrightarrow_{\mathrm{df}} x$ is a natural number and $F(x)$.

[3] A good example of implicit stipulative definition is ZF set theory, as a sort of explication of the concept of "set."

[4] I suppose that there is no redundant component of the definiens, such as $F(x) =_{\mathrm{df}} x$ is a natural number and ($F(x)$ or not $F(x)$)

It entails that "$F(a)$" is true if and only if a is a natural number and $F(a)$. Let a=3. Is the sentence "$F(3)$" true or not? The definition does not help us select the right answer, because we do not know the extension of the "F" predicate. Note that some interpretations satisfy the open formula "$F(x) \leftrightarrow x$ is a natural number and $F(x)$," and some do not; however, no contradiction follows from the definition. On the other hand, it is a mistaken definition, because it is circular.

iii. Missing existential condition

This signifies that the definition entails a logical contradiction. Therefore, the predicate or function that the definition is intended to introduce does not exist by reductio, because contradiction follows from its supposed existence (many philosophers mistakenly refer to paradox only as this kind of definition). For example, $F(x) \leftrightarrow_{\mathrm{df}} \sim F(x)$. Hence, "$F(a) \leftrightarrow \sim F(a)$" is false in any interpretation of the "F" predicate and "a" individual constant. As there is only an atomic formula on the left-hand side and the definition is only one line, the definiendum must also occur on the right-hand side; otherwise, there is no way to infer a logical contradiction from the definition. Accordingly, if contradiction (or tautology) follows from a definition schema: "$F(x) \leftrightarrow_{\mathrm{df}} \ldots$," then it is necessarily circular.

We have now reached the point of defining a few simple sequences.

2.1.1. Consider the next endless sequence of formulas:

"$S(1) \leftrightarrow \sim S(1)$," "$S(2) \leftrightarrow \sim S(2)$," "$S(3) \leftrightarrow \sim S(3)$," … "$S(n) \leftrightarrow \sim S(n)$"

Such a formula sequence can be changed to a natural language sentence sequence by applying a translation key to the natural language in the following way: $S(n)$ =: n is a prime number. In fact, every sequence is a function, namely, a set of ordered pairs, the first element of which is a natural number – an ordinal number that defines the place in the sequence. The second element is the value of the function at that ordinal number.

Our current example can be formulated in a set theoretical language without the ellipsis in the following manner:

$\{\langle n, \ulcorner S(n) \leftrightarrow {\sim}S(n)\urcorner \rangle : n \in \acute{\omega}\}$ where $\acute{\omega}$ is the set of natural numbers and $\ulcorner\urcorner$ is the citation function.[5]

If we were to construct truth-value sequences instead of sentence sequences, then the result would be a very dull function that relates "false" logical value to every natural number. A more interesting example is another sequence of sentences: $3\neq1$, $3\neq2$, $3\neq3$, $\ldots 3\neq n$ (for the sake of simplicity, the quotation marks have been omitted). The following is the defining formula of the sequence without the ellipsis:

$\{\langle n, 3\neq n\rangle : n \in \acute{\omega}\}$

2.1.2. Let us formulate the previous example in a more general term.

$A(n) \leftrightarrow_{\mathrm{df}} {\sim}B(n)$ where $n \in \acute{\omega}$

or by applying a set theoretic language, we get:

$\{\langle n, {\sim}B(n)\rangle : n \in \acute{\omega}\}$

The translation of "B" predicate has been presented in example 2.2.1. – $B(n)$ =: $3=n$. In the electronic version of the examples, the interpretation of $A(n)$ formula sequence is more interesting.

2.1.3. There is no problem with the following sequence of formulas as well:

$\{\langle n, S(n)\rangle : S(n) \leftrightarrow {\sim}S(n)\}$

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[5] In the present paper it is not important whether zero is a natural number.

The above-mentioned formula defines the empty set in a rather verbose manner, because the condition "$S(n) \leftrightarrow {\sim}S(n)$" is a logical fallacy.

2.1.4. But what should we make of the following example?

$S(n) \leftrightarrow_{\mathrm{df}} {\sim}S(n)$ where $n \in \acute{\omega}$

or in a set theoretic language:

$\{\langle n, |S(n)|\rangle : {\sim}S(n) \; \& \; n \in \acute{\omega}\}$
$|S(n)|$ is the truth value of $S(n)$

Both the above-mentioned definitions are wrong, because the existence condition does not hold, and hence, it is questionable whether it is a meaningful set definition at all, and it is not obvious what the elements of the set are, if any.

What is the difference between a sentence (1) and a definition (2)?

(1) $\varphi_1(\Lambda) \leftrightarrow {\sim}\varphi_1(\Lambda)$

(providing that "$\varphi_1(\Lambda)$" formula has an interpretation).
and

(2) $\varphi_1(\Lambda) \leftrightarrow_{\mathrm{df}} {\sim}\, \varphi_1(\Lambda)$

Consider that the contradiction that "$\varphi_1(\Lambda)$" is true. From (2), "${\sim}\varphi_1(\Lambda)$" is true. Therefore, "$\varphi_1(\Lambda)$" is untrue, whence contrary "$\varphi_1(\Lambda)$" is false.
If "$\varphi_1(\Lambda)$" was false, then according to (2), "${\sim}\varphi_1(\Lambda)$" would be false; if so, then "$\varphi_1(\Lambda)$" is not false, whence "$\varphi_1(\Lambda)$" is true after all. Thus, "$\varphi_1(\Lambda)$" is true and false.

Thus, (2) is a paradox, but what about (1)? Is it a valid argumentation based on (1)? Let us consider example 2.1.1.

2.1.5. The following definition of a set is a recursive definition:[6]

$S(1)$ =: Socrates was wise.

$S(n) \leftrightarrow_{\mathrm{df}} \sim S(n-1)$ where $n > 1$ and $n \in \acute{\omega}$

$\{ \langle 1, |S(1)| \rangle, \langle n, |S(n)| \rangle : \sim S(n-1) \ \& \ n > 1 \ \& \ n \in \acute{\omega} \}$

As long as we rely on Plato's judgment of Socrates, this sequence starts in this manner: True, false, true, false, true, false, …

If the base of the above-mentioned recursive definition is missing, that is, the definition of $S(1)$ is missing, then the above-mentioned definition would be circular, and would fail to define the set.

2.1.6. If we restrict the natural numbers to a finite subset, we can apply a reverse version of the recursive definition in 2.1.5 (in the case of an endless sequence of the recursive series, there is no way to invert the sequence, because there is no greatest natural number). Let us look at this for the first five natural numbers:

$S(n) \leftrightarrow_{\mathrm{df}} \sim S(n+1)$ where $n \in \{1,2,3,4\}$

$S(5)$ =: Socrates was wise.

$\{ \langle 5, |S(5)| \rangle, \langle n, |S(n)| \rangle : \sim S(n+1) \ \& \ n \in \{1,2,3,4\} \}$

Thus, we get the beginning of the previous sequence: true, false, true, false, true.

[6] Every recursive definition must have a base case(s) (basis) and inductive clause, and most of them an external clause. "The difference between a circular definition and a recursive definition is that a recursive definition must always have base cases, cases that satisfy the definition without being defined in terms of the definition itself, and all other cases comprising the definition must be 'smaller' (closer to those base cases that terminate the recursion) in some sense. In contrast, a circular definition may have no base case, and define the value of a function in terms of that value itself, rather than on other values of the function. Such situation would lead to an infinite regress." from Wikipedia, the free Internet encyclopedia.

2.1.7. In the spirit of Goldstein's illuminating example, let us omit the greatest element from the previous example. We then get the next mistaken definition range over the natural numbers:

{ $\langle n, |S(n)|\rangle$: ~$S(n+1)$ & $n \in \acute{\omega}$ }. For natural language, this means:

(1) The next sentence is false.

(2) The next sentence is false.

(3) The next sentence is false.

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We are not able to choose an assignment for the members of the sequence that results in true sentences only – therefore, there is no model of the sequence, but the following interpretation does not produce antinomy: true= |(1)|, false= |(2)|, true= |(3)|, false= |(4)|, …

The following interpretation is also consistent:

false= |(1)|, true= |(2)|, false= |(3)|, true= |(4)|, …

We do not know what to select, because the definition of the last member of the sequence is missing.

In fact, it is a formula sequence without translation into natural language, but we may interpret "*S*" unary predicate letter in the following way: *n* – has a hat on. In this case, men standing in the line have and do not have a hat on, in turn; however, we do not know whether the even serials are with hats or bareheaded. If we formulate the definition in formal language, then it is clear that "*S"* predicate letter does occur on both the sides of the definition, and hence, it is circular. The existence condition of the sequence is not false, because we have two consistent interpretations of the formulas; however, the definition does not help us select the only valid interpretation, and hence, it is mistaken.

2.1.8. Notice that the following modification of Yablo's sequence does not produce antinomy in any way:

(1) For every $k>1$, S_k is not true

(2) For every $k>2$, S_k is not true

(3) For every $k>3$, S_k is not true

.

.

There is no trouble with this sequence because the numbers, which act as the names of sentences, are not constituents of the sequence itself. In fact, this is not a meaningful sentence sequence, but a formula sequence, which can be transformed into a sentence sequence by applying a feasible interpretation of S_k predicate. We can devise such an interpretation that every member of the sentence sequence will be true, and hence, there will be a semantic model of the sentence sequence. (A) Let S_k be defined in the range of natural numbers, such that S_k, if and only if $k=1$. By applying this interpretation, the result is: (1) is true, (2) is true, (3) is true, …. (n) is true. (B) If we select another interpretation, such that S_k if and only if $k=2$, then (1) would be false, but the rest would be true. Accordingly, the definitions of the specific sequences, where n is a natural number, are as follows:

(A) $S(n) \leftrightarrow_{\mathrm{df}}$ for every $k>n$, and k is not equal to 1 (in other words, there is no such $k>n$, and that $k=1$).

(B) $S(n) \leftrightarrow_{\mathrm{df}}$ for every $k>n$-re, and k is not equal to 2 (in other words, there is no such $k>n$, and that $k=2$).

Note that neither definition (A) nor definition (B) is circular. Hence, $S(n)$ sentence sequence is not paradoxical, and every member of the sequence has one and only one truth value; furthermore, we can select an interpretation in which every member of the sequence is true, so that it has a model (it is worth studying this problem on the electronic model).

2.1.9. Let us alter the previous definition to a recursive one as follows:

$S(1)$ =: Socrates was wise.

$|S(1)|$ = |Socrates was wise.| = true

$S(n) \leftrightarrow_{\text{df}} \forall k\ (k < n \rightarrow \sim S(k))$ where $1 < n$ and $n \in \omega$

$\{\langle 1, |S(1)|\rangle, \langle n, |S(n)|\rangle: \forall k\ (k<n \rightarrow \sim S(k))\ \&\ 1<n\ \&\ n \in \omega\}$

In this case, the sequence of the sentences results in the following truth value sequence:

⟨true, false, false, false, false, …⟩

What if Socrates was not wise?
(Ask the electronic model).

2.1.10. If we restrict the sequence to many finite members, then we can define an inverse recursive sequence as follows:

$S(n) \leftrightarrow_{\text{df}} \forall k\ (n < k \rightarrow \sim S(k))$ where $n < 6$ and $n \in \omega$

$S(6)$ =: Socrates was wise.

$|S(6)|$ = |Socrates was wise| = true

$\{\langle 1, |S(6)|\rangle, \langle n, |S(n)|\rangle: \forall k\ (n<k \rightarrow \sim S(k))\ \&\ n<6\ \&\ n \in \omega\}$

We will obtain a reverse finite segment of the previous sequence as follows:

⟨false, false, false, false, false, true⟩

Let us omit the greatest element of the sequence. We will then get Yablo's sequence as follows:

$\{\langle n, |S(n)|\rangle: \forall k\ (n < k \rightarrow \sim S(k))\ \&\ n \in \omega\}$

The electronic model presents the consequence in the event that there is no greatest elements of this reverse recursive sequence. If we delete it from the electronic sheet, then we can observe the result.

Finally, by applying our results, we can compare three earlier mistaken definitions as follows:

$\{\langle n, |S(n)|\rangle : {\sim}S(n) \ \&\ \mathrm{n}{\in}H\}$

$\{\langle n, |S(n)|\rangle : \forall k\ (n < k \rightarrow {\sim}\ S(k))\ \&\ n{\in}H\}$

$\{\langle n, |S(n)|\rangle : {\sim}S(n{+}1)\ \&\ n{\in}H\}$

If H is the infinite set of natural numbers, then the existence condition of the first two definitions does not hold and we can observe that the base of the second two reverse recursive definitions is missing. If $\langle H,<\rangle$ binary relation does have a greatest element, then the existence condition of the second two definitions holds, but that of the first definition does not hold in any assignment of the non-empty H set.

2.2. Investigation of three second-order logic formulas

The first second-order formula asserts the existence of a specific function, the range of which consists of well-formed formulas. They fulfill all the syntactical criteria for being sentences. The second formula claims the existence of an arithmetical function, and the third states that there is an expedient sequence of true or false sentences. No doubt the first function exists with a range of well-formed formulas, but a paradox arises if we suppose that the existence of the first function supports the existence of a second and third function. The antinomy is that contradiction, and therefore falsity, follows from the existence of the second and third functions, while, on the contrary, there is no problem with the first function, which seemingly proves the existence of the second and third functions.

In the original interpretation of Yablo's series, the domain of the sequence is isomorphic with the structure of natural numbers. Let us consider the interpretation changes with the domain of Yablo's sequences as set H. We can take H as the freely interpretable input of the formula that generates the list. Sometimes, we may suppose that H is isomorphic with natural numbers and has a least element; sometimes, we may consider that it is an infinite set of integers, but has one and only one maximal element; and sometimes, we may suppose that H is finite. We can refer to (AR) when a line in the inference is a consequence of the $A1$, $A2$, or $A3$ axioms.

$A1$. Let "<" be a binary, asymmetric, and transitive relation that defines a strict linear order over the domain of H.
$A2$. If $n\in H$ and n is not equal to the greatest element of H, then n has a unique successor, that is, $n+1$, and $(n+1)\in H$. This means that there is no z in H such that $n<z$ and $z<n+1$.
$A3$. In some cases, we can suppose that there is a successor of n for every $n\in H$.[7]
Now, let us consider the next three second-order logic formulas.

2.2.1. The first function:

(Ya) $\exists f\forall n$: $n\in H\rightarrow f(n) = \ulcorner \mathrm{S}(n) \leftrightarrow \forall k\ (n < k \rightarrow {\sim}\mathrm{S}(k))\urcorner$

The function defined by (Ya) is f, given as follows:

$f =_{\mathrm{df}} \{<n, \ulcorner S(n) \leftrightarrow \forall k\ (n < k \rightarrow {\sim}S(k))\urcorner > : n\in H\}$

Note the citation function $\ulcorner\ldots\urcorner$ in the definiens. The initial segment of the function claimed by (Ya) is the following list:

$f(0)$ = "$S(0) \leftrightarrow \forall k\ (0 < k \rightarrow {\sim}S(k))$"
$f(1)$ = "$S(1) \leftrightarrow \forall k\ (1 < k \rightarrow {\sim}S(k))$"
$f(2)$ = "$S(2) \leftrightarrow \forall k\ (2 < k \rightarrow {\sim}S(k))$"
…

[7] Jeffrey Ketland pointed out that Yablo's paradox arises if "<" transitive and $\forall x(x\in H\rightarrow\exists y(y\in H\ \&\ x<y))$ (Ketland, 2004).

The domain of the function is the set of natural numbers – or a structure isomorphic with it – and its range is a denumerable set of well-formed formulas. Note that if we apply Tarski's T-schema for any member of the list, then we get an interpreted formula, a sentence that is false (contradiction). The definition of the function is not circular, and subsequently not mistaken. To prove this, for the sake of simplicity, we can introduce the following notation:

$G(n) =_{\text{df}} S(n) \leftrightarrow \forall k\ (n < k \rightarrow \sim S(k))$

By applying this notation, we can get a simpler sequence of truth values, $\{\langle n, |G(n)|\ \rangle : n \in H\}$, which is not a circular definition of a function because no part of the definiendum occurs in the definiens. The result is a dull constant function, ⟨0, false⟩,⟨1, false⟩,⟨2, false⟩, …

2.2.2. In the second case, consider an arithmetical function defined in such a way that it satisfies the following condition:

$(Yb)\ \exists f \forall n\text{: } n \in H \rightarrow (\text{odd}=f(n) \leftrightarrow \forall k\ (n < k \rightarrow \text{even}=f(k)))$

The sequence claimed by (*Yb*) is *Y*, which can be given as follows:

$Y = \{\langle n, f(n)\rangle : \text{odd}=f(n) \leftrightarrow \forall k\ (n < k \rightarrow \text{even}=f(n))\ \&\ n \in H\}$

Suppose that *H* is a set of natural numbers. The initial segment of the presupposed function asserted by (*Yb*) can be given as follows:

$\langle 0, f(0)\rangle, \langle 1, f(1)\rangle, \langle 2, f(2)\rangle, \ldots$

The contradiction then arises following the argumentation:

*(1) odd=$f(n)$ (Yb) f if the value of f is odd for n

*(2) $\forall k.\ n<k\rightarrow$ even=$f(k)$ (Yn)

*(3) $n<n+1\rightarrow$ even=$f(n+1)$ (2)

*(4) $n<n+1$ (AR)

*(5) even=$f(n+1)$ (3) (4)

*(6) ($\forall k.\ n<k\rightarrow$ even=$f(k)$)$\rightarrow$($\forall k.\ n+1<k \rightarrow$ even=$f(k)$) (AR)

*(7) $\forall k.\ n+1<k \rightarrow$ even=$f(k)$ (2) (6)

*(8) odd=$f(n+1)$ (7) ($Yn+1$)

(9) odd=$f(n)$ $\rightarrow$. even=$f(n+1)$ and odd=$f(n+1)$ *(1) (5) (8)

(10) even=$f(n)$ (9) (from elementary arithmetic)

(11) $\forall n$. even=$f(n)$ (10) by universal generalization over n in (10), as n was arbitrary.

(12) $\forall k.\ n<k\rightarrow$ even=$f(k)$ (11) (AR)

(13) odd=$f(n)$ (Yn) (12)

(14) even=$f(n)$ and odd=$f(n)$ (10) (13)

As there is no such number that is even and odd, (14) is therefore a contradiction, and hence, the function f does not exist and (Yb) is false. Yablo's antinomy is often called a ω paradox, because f function does exist for any large finite initial segment of Yablo's sequence, while in the case of an endless sequence, f does not exist and Y=∅. By matching the property of being odd to truth, and the even property to a false truth-value, we can get the third second-order logic formula. Let $\sim S(n)$ =: even=$f(n)$ and $S(n)$=: odd=$f(n)$.

2.2.3. Consider the affirmation of an endless sentence list as follows:

(Yc) $\exists s\forall n$: $n\in H \rightarrow (s(n) \leftrightarrow \forall k(n < k \rightarrow \sim s(k)))$

The sequence claimed by (Yc) is:

$\{\langle n, |S(n)| \rangle : \forall k\ (n < k \rightarrow \sim S(k))\ \&\ n\in H\}$

Let us apply the following notation:

$G(s, n) =: \forall k\ (n < k \rightarrow {\sim}s(k))$

By putting into practice, the above-mentioned notation (*Yc*) turns into a simpler formula:

(*YPG*= Yablo's paradox generator) $\exists s \forall n$: $n \in H \rightarrow (s(n) \leftrightarrow G(s, n))$

(2) $\forall n$: $n \in H \rightarrow (S(n) \leftrightarrow G(S, n))$ (*YPG*) *S*

Thus, we can obtain the following set:

$\{\langle n, |S(n)|\ \rangle : G(S,n)\ \&\ n \in H\}$

In this schema, as reverse recursive definition without base case is circular, the definiens – $G(S, n)$ – comprise some part of the definiendum – $S(n)$ – that is, "*S*" functor, which has been referred to in the Introduction.

Based on the creative application of (*YPG*) schema, one can construct plenty of paradoxes. Recently, Roy T. Cook presented two interesting applications of the (*YPG*) schema:

$G_1(s, n) =_{\mathrm{df}} \forall k\ ((n < k \rightarrow s(k)) \rightarrow \Phi)$

$G_2(s, n) =_{\mathrm{df}} \forall k\ (n < k \rightarrow (s(k) \rightarrow \Phi))$

Cook proved that, similar to the Curry paradox, every Φ sentence is derivable from (*YPG*) replacing G with G_1 or G_2. (Cook, 2009)[8] This is not very surprising, because in the framework of classical logic, everything follows from falsity, and (*Yc*) sentence is false.

This can readily be proved as follows:

[8] See my comments: http://www.andrasek.hu/ferenc/papers/cook9.doc

Suppose H is the set of natural numbers. Then, the initial segment of the sequence generated by (Yc) is as follows:

($Y0$) $S(0) \leftrightarrow_{\mathrm{df}} \forall k.\ 0<k \rightarrow \sim S(k)$

($Y1$) $S(1) \leftrightarrow_{\mathrm{df}} \forall k.\ 1<k \rightarrow \sim S(k)$

($Y2$) $S(2) \leftrightarrow_{\mathrm{df}} \forall k.\ 2<k \rightarrow \sim S(k)$

…

(Yn) $S(n) \leftrightarrow_{\mathrm{df}} \forall k.\ n<k \rightarrow \sim S(k)$

($Yn+1$) $S(n+1) \leftrightarrow_{\mathrm{df}} \forall k.\ n+1<k \rightarrow \sim S(k)$

It was this sequence that Stephen Yablo invented. Note that (Yc) is obviously false if we assign s variable for the predicate "to be equal to itself" and if H is the set of natural numbers.

(1) $\forall n$: $n \in \omega \rightarrow (n=n \leftrightarrow \forall k\ (n < k \rightarrow \sim n=n))$

(2) $1 \in H \rightarrow (1=1 \leftrightarrow \forall k\ (1 < k \rightarrow \sim 1=1))$ (1)

(3) $1 \in H \rightarrow (1=1 \leftrightarrow (1 < 2 \rightarrow \sim 1=1))$ (2)

(4) $T \rightarrow (T \leftrightarrow (T \rightarrow F))$ (3) (from elementary arithmetic)

(5) F (4)

It is more complex to show that it is false to assign with any other predicate, such as "have a hat on."[9] The contradiction arises as follows:

*(1) $S(n)$ - Suppose $S(n)$ is true (Yc) S

*(2) $\forall k.\ n<k\rightarrow \sim S(k)$ (Yn)

*(3) $n<n+1\rightarrow \sim S(n+1)$ (2)

*(4) $n<n+1$ (AR)

*(5) $\sim S(n+1)$ (3) (4)

*(6) $(\forall k.\ n<k\rightarrow \sim S(k))\rightarrow(\forall k.\ n+1<k \rightarrow \sim S(k))$ (AR)

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[9] I will tell a story illustrating the problem. It is somewhat similar to Roy A. Sorensen's and Graham Priest's version of the paradox. In heaven, St. Peter is responsible for operations. Heaven is remarkably large but not limitless, and St. Peter has to ensure that it never becomes full. The only thing that St. Peter must do is to ensure that he never allows an infinite queue to get in. He is aware that the devil is always trying to find out how to set up an endless queue at the gates of heaven. However, sometimes, there are quarrels at the gate. Therefore, St. Peter introduces strict rules eliminating both problems:

a. everyone has one and only one line number in sequential order; and

b. everyone has a hat on if, and only if, nobody behind them has a hat on.

St. Peter has done his job well; he does not have to bother with the gate if there is an endless queue before the entrance. In the event that the queue is finite, St. Peter does have a job to do, which depends on the people in the queue complying with the rules. If there is only one soul before the entrance, that soul will argue as follows: If there is someone with a hat on behind me, then I have no hat on; but nobody is standing behind me, so I have to have my hat on. What if there are two souls before the gate? The second one puts forward a similar argument, so has a hat on. The first one knows there is somebody behind him or her with a hat on, so he or she does not have a hat on. The situation with three souls is easy to determine. This can be illustrated as follows:

X
OX
OOX

Is the next figure a possible situation according to the rules?

OXX

No, because the second soul has a hat on only in the event that there is nobody with a hat on behind him or her. How can we formulate the general problem in the language of formal logic? Suppose every soul in heaven is represented by an ordinal number, the question is then: Is there a function f that satisfies the following condition?

For every n natural number (hatted=$f(n)$ iff for every k (if $n<k$, then not hatted=$f(k)$))

What if there is an endless queue before the gate? I will prove that there is no such f function in the domain of natural numbers. Let us consider, on the contrary, that there exists a function f_1 that satisfies the conditions. In this case, n has a hat on or not. Consider the first: hatted=$f_1(n)$. If n has a hat on, then everybody behind him or her is bareheaded. Yes, but then, soul $n+1$ close behind him or her also has a hat on. However, if he or she has a hat on, then the soul in front of him or her has no hat on, and hence, n is bareheaded: not hatted=$f_1(n)$. We have a contradiction; thus, our premise was wrong, and therefore, n soul must be bareheaded. However, this is true for every soul in the queue; thus, nobody has a hat on. Nevertheless, anyone is permitted to have a hat on, which is again a contradiction. In this argument, f_1 is arbitrary, and thus, we have proved that no f function exists that fulfills the conditions.

*(7) $\forall k.\ n+1<k \rightarrow \sim S(k)$ (2) (6)

*(8) $S(n+1)$ (7) (Yn+1)

(9) $S(n) \rightarrow .\ \sim S(n+1)$ and $S(n+1)$ *(1) (5) (8)

(10) $\sim S(n)$ (9)

(11) $\forall n.\ \sim S(n)$ (10) because n was arbitrary

(12) $\forall k.\ n<\mathrm{k} \rightarrow \sim S(k)$ (11) (AR)

(13) $S(n)$ (Yn) (12)

(14) $\sim S(n)$ and $S(n)$ (10) (13)

(15) If $\exists s \forall n$: $n \in H \rightarrow (s(n) \leftrightarrow \forall k\ (n < k \rightarrow \sim s(k)))$, then $\sim S(n)$ and $S(n)$ (Yc) (14)

(16) $\sim \exists s \forall n$: $n \in H \rightarrow (s(n) \leftrightarrow \forall k\ (n < k \rightarrow \sim s(k)))$ (15)

In other words, this means $\{\langle n, |S(n)|\ \rangle : \forall k\ (n < k \rightarrow \sim S(k))\ \&\ n \in \mathrm{H}\} = \varnothing$, and hence, the $S(n)$ sequence does not exist. It is important that the contradictions disappear if:

a. H = negative integers
b. we extend the domain of (Yb), (Yc) functions to the first transfinite ordinal (ω) that is greater than any natural number ($H = \omega \cup \{\omega\}$)
c. H = positive and negative integers $\cup\ \{\omega\}$
d. we apply inverse relation of "<" relation.

This signifies that it is not the concept of infinity that is the root of the problem. Note that we have not applied any version of T-schema, omega rule, or fixed-point construction in this ontologically and formally parsimonious method of deductive inference. The above-mentioned derivation therefore refutes J. Ketland's claim: "The derivation of an inconsistency requires a uniform fixed-point construction. Moreover, the truth-theoretic disquotational principle required is also uniform, rather than the local disquotational T-scheme. The theory with the local disquotation T-scheme applied to individual sentences from the Yablo list is also consistent" (J. Ketland, 2005). The antinomy demonstrates, as a proof by contradiction, that only such a series of definitions exists, and not the assignment of true or false to those definiendums as sentences. A map of definiendums followed by consistent evaluation, known as an “evaluation map” (and not “model”) does not exist in the framework of classical logic. However, if someone inquires as to the truth value of (*Yc*) second-order logic formula, its truth value is falsity, and thus, it has no model, as demonstrated by this antinomy. Let us now turn to a summary table of results from the analysis of (*Ya*), (*Yb*), and (*Yc*) formulas. Let us consider an *x* term of a formula as “input of a formula” iff *x* term (a predicate, function, sentence letter, or individual constant) is freely interpretable in a certain context. In other words, a term with a rigid meaning is not an input of a formula.

The domain of the evaluation map (*H* set) is:	The evaluation map of:		
	(*Ya*)	(*Yb*)	(*Yc*)
Finite	Exists		
Denumerable infinite, and has no greatest element	Exists	Does not exist	Does not exist
Denumerable infinite, and has a greatest element	Exists		

3. Conclusion

Let us return to the (*Yc*) formula:

$$\forall n\text{: } n \in H \rightarrow (S(n) \leftrightarrow_{\text{df}} \forall k(n < k \rightarrow {\sim}S(k)))$$

If *H* has no greatest element, then the $S(n)$ sequence is not well founded, and therefore, the above-mentioned formula, as a definition of $S(n)$, is mistaken, because the definiens – $\forall k(n<k \rightarrow {\sim}S(k))$ – comprises an element of the definiendum, that is, “*S*.” Thus, Priest was right from this point of view, but only Goldstein suspected the reason. The definition is circular because there is no largest natural number. Thus, a mistaken reverse recursive definition generates Yablo’s sequence, which has no base case. Moreover, its existence condition fails in the domain of natural numbers.

It is erroneous to define a sequence using a reverse recursive definition without a base case, just as it is false to presuppose the existence of a non-existent sequence. There is a selection of set *H* – the domain of the sequence – when the existence condition holds, and hence, the paradox is solvable. The solution is that set *H* has not only a least, but also a greatest, element that is the base case of the reverse recursive definition. As a result, Yablo's paradox resembles Buridan's paradox rather than the Liar paradox.[10] This can be explained in detail as follows:

Philosophers understand paradoxes from two points of view. The first type of understanding seeks a cure for the disease (e.g., Tarski's meta language – object language distinction or Kripe's theory of truth); the second, as it takes only a note of the disease, merely requires simulating the trouble and making an adequate model of the disease. (e.g., Dialetheism, The Revision Theory of Truth; combinational circuits can simulate truth functions and sequential circuits can simulate semantic paradoxes. The

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[10] For those who do not know Buridan's paradox:
"Twelfth sophism: GOD EXISTS AND SOME CONJUNCTION IS FALSE
The twelfth sophism is 'God exists and some conjunction is false'.
Let us posit that this is written on the wall and that there exists no other proposition than it and its parts. And then it is asked whether it is true or false.
We argue as before: for if it is true, then it follows that it is false; and if it is false, it seems to follow that it is true, for things are as it signifies, since its contradictory is false, namely this: 'God does not exist or no conjunction is false'.
Solution: we should say that it is false, and the argument is solved as before. For although things are as it signifies according to its formal signification, yet, things are not as would be signified by the consequent implied by it and the case, and, assuming it to be named by the proper name A, its contradictory would be this: 'No God exists or no conjunction is false or A is not true'.
Similar sophisms could be formed concerning disjunctive propositions, as 'A man is a donkey or some disjunctive is false', positing that there is no other disjunctive; and the same goes for exceptive [propositions], as for example, 'Every proposition other than an exceptive is true', positing that there are no propositions except this exceptive and two others, namely, that God exists and that a man is an animal; and thus also with exclusives, as when Socrates says: 'God exists' and Plato says: 'Only Socrates says something true', and nobody says anything else. Other sophisms can also be formed about the fact that it is possible for a proposition to be doubtful or not doubtful, known, or not known, believed or not believed."
John Buridan, Summulae de Dialectica (Summulae), an annotated translation with a philosophical introduction by Gyula Klima, New Haven: Yale University Press, 2001, Sophismata, c. 8, p. 980.

AND or NAND gates are combinational digital circuits, while Flip-flops and registers are sequential circuits.)[11] The latter approach often leads to the amelioration of the patient, and is sometimes, a source of effective therapy. Another question that has been briefly addressed is whether the source of the paradox is a solitary wrong definition, a mistaken language, or both of these. We can come to a clearer understanding of the concept of paradox if we explain its usage in the context of related concepts. Let a set of formulas of a formal logic language *L* be defined. We are talking about an evaluation map of a subset of *L* formulas if we define a certain function with the domain of formulas and range of truth values; about the models of the sets of formulas if there is an evaluation map that considers every formulas of the set to be true; and about a certain formal language that allows some terms to be formulated and forbids the others. The following table – as an example – is self-explanatory: "$\ulcorner\urcorner^1$" is a name-forming functor; φ_1 is the inverse of "$\ulcorner\urcorner^1$" functor (other functors are "$\ulcorner\urcorner^2$" and "φ_2"); "1,0" are truth values – true and false, respectively; "*T*" is the "true" predicate; Λ is the name of the Liar sentence; τ is the name of the truth-teller sentence; *B* is the name of a sentence of the medieval French philosopher Jean Buridan; and *G*="God exists."

[11] "In digital circuit theory, sequential logic is a type of logic circuit whose output depends not only on the present input but also on the history of the input. This is in contrast to combinational logic, whose output is a function of, and only of, the present input. In other words, sequential logic has storage (memory) while combinational logic does not." From Wikipedia, the Free Encyclopedia

Set of formulas	Context	Evaluation map	Model *
$\{p, \sim q\}$	First-order logic	Defined for all four cases: $<0=\vert p\vert, 0=\vert q\vert> \to <0,1>$ $<0=\vert p\vert, 1=\vert q\vert> \to <0,0>$ $<1=\vert p\vert, 0=\vert q\vert> \to <1,1>$ $<1=\vert p\vert, 1=\vert q\vert> \to <1,0>$	There is: $1=\vert p\vert$, $0=\vert q\vert$
$\{ \varphi_1(\Lambda)\leftrightarrow \sim \varphi_1(\Lambda) \}$	First-order logic	Defined for both cases. $<0=\vert\varphi_1(\Lambda)\vert> \to <0>$ $<1=\vert\varphi_1(\Lambda)\vert> \to <0>$	Does not exist
Buridan paradox: $\{\underline{\varphi}_1(B) \leftrightarrow_{df} \sim(\varphi_1(G) \vee \varphi_1(B)) \}$	First-order logic	Evaluation map is a partial function because B is not an input of the formula. The definition is circular. $<1=\vert\varphi_1(G)\vert> \to <0>$	Exists: $1=\vert\varphi_1(G)\vert$ In case of $0=\vert\varphi_1(G)\vert$, contradiction arises from the definition.
Liar sentence: $\sim \varphi_1(\Lambda) =: \Lambda$ sentence is false. $\{\varphi_1(\Lambda)\leftrightarrow_{df} \sim \varphi_1(\Lambda)\}$	First-order logic	Evaluation map does not exist because there is no input of the formula. The definition is circular.	Does not exist; contradiction arises from the definition.
Strengthened Liar: $\{\Lambda =_{df} \ulcorner \sim T(\Lambda) \urcorner^1, T$ schema$\}$ $\Rightarrow \sim\Lambda =_{df} \ulcorner \sim T(\Lambda) \urcorner^1$	A closed language, in terms of Tarski	Comment: the language is unsound or $\lambda s. \ulcorner s \urcorner^1$ name-forming function is not permissible.	Does not exist; contradiction arises from the definition.
Truth-teller paradox: (τ) τ sentence is true. $\{\varphi_1(\tau)\leftrightarrow_{df} \varphi_1(\tau)\}$	First-order logic	Evaluation map does not exist because there is no input of the formula. The definition is circular.	Does not exist; tautology arises from the definition.
Yablo's paradox: $\{\exists s\forall n{:}n\in H\to(s(n)\leftrightarrow\forall k(n<k\to\sim s(k)))\}$	Second-order logic	$<H$ is finite$>\to <1>$ $<$H = natural numbers$>\to <0>$ $<$i has the greatest element$>\to<1>$	Exists, if H has a greatest element.
Yablo's paradox: $\forall n{:}n\in H\to(S(n)\leftrightarrow_{df}\forall k(n<k\to\sim S(k)))$ $\{\langle n,\vert S(n)\vert\rangle{:}\forall k(n<k\to\sim S(k))\&n\in H\}$	First-order logic	Evaluation map is a partial; the definition is mistaken. $<H$ is finite$>\to <0 \ldots 01>$ $<H$ has the greatest element$=$ $H>\to<0\ldots01>$	Exists, if H has only one element.

* By applying finite formal logic language fragments, every formula and its evaluation map can be simulated in the spreadsheet software (e.g., Excel).[12]

Let us explain the above-mentioned table. The paradox is not a lonely warrior, but a soldier in an army. In certain language contexts, an H set of formulas is not paradoxical if in any of the non-empty domain of discourses, any interpretation of inputs (predicates, functions, and names), and any evaluation map of sentence letters, and

[12] See my spreadsheet models: http://www.andrasek.hu/ferenc/papers.htm

consistently and unequivocally assigns the truth values to all the sentences in *H*. On the contrary, if:

1. every evaluation map of the sentence letters;
2. every interpretation of predicates, functions, and individual names; or
3. every non-empty domain of discourse,

does not assign consistently and unequivocally the truth values to all the sentences, then we are denoting the logical paradox.

(In the case of Yablo's paradox, this means that if only one sentence in the sentence series is ambiguous in a certain selection of the domain of discourse, then the whole series is paradoxical). A paradox is partially solvable if the evaluation map partially ensures the truth values of the formulas (e.g., Buridan's paradox). However, it is unsolvable if there is no consistent evaluation map at all (e.g., the Liar paradox), or it has no definite truth value (e.g., 2.1.7. Example). The paradox is eliminable if we know how to block the composition or derivation of the paradox, by applying a specific formal logic framework. A semantic paradox cannot be formulated in the framework of classical logic, and thus, we have to apply a mistaken semantically closed formal logic in the case of the Strengthened Liar.

This is the penultimate sentence of this paper, which is true if and only if none of the foregoing sentences is false. "The King Has No Clothes!" [13]

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