What is demonstrated by Yablo's paradox?

In the formation of his paradox Stephen Yablo introduces the following sentence:

(S1) for all $k > 1$, Sk is untrue

Let us ask what does the expression, 'for all k >1, S*k* is untrue', mean? To answer this, 'S*k*' has to be the name of a sentence because 'is untrue' is a semantic predicate. What about S1? It too is the name of a sentence, that is, the sentence, 'for all k > 1, S*k* is untrue'. Thus, both S1 and S*k* are not sentences but rather names of sentences and the introduction of quotation marks or other name forming device are required. For example,

S1 = 'for all *k* >1, S*k* is untrue'
S2 = 'for all *k* >2, S*k* is untrue'
or
S1 = ⌜for all *k* >1, S*k* is untrue⌝
S2 = ⌜for all *k* >2, S*k* is untrue⌝

However both are problematic, better to eliminate. We shall demonstrate that Yablo's paradox may be formulated without the use of semantic predicates, sentence variables and the quotation marks or name forming devices these imply. In order to formulate Yablo's paradox without them, let *S* be a one-one injective map from a subset of integers to sentences. The integers will also act as names of sentences. So the map may also be construed as a one-one injective map from sentence names to sentences. From grammatical aspect this map is a unary predicate. On the other hand S is inverse of the citation-function or other name forming device, this is why we call 'function' and not 'predicate'. Yablo's sequence may be reformulated as

(S1)* For all $k > 1$, k is untrue

Note the absence of quotation marks here, which is correct, because *k* is a sentence name and not a sentence. Using a simplified version of Tarski's T-schema

(T) *X* is true iff *p* where *X* is the name of the object language sentence *p*

we obtain

k is untrue ↔ ~S(k)

and hence

(S1)** For all k >1, ~S(k)

The following is a fragment of an instance of the map *S* together with a putative evaluation of the sentences to which *S* maps the sentence names (integers).

k	*S*(k)	Evaluation
0	Snow is not white.	False
-1	Snow is white.	True
1	Socrates is not wise.	False
-2	Socrates is wise.	True
2	There is somebody who is undying.	False
-3	All men are mortal.	True

However, we shall soon see that a consistent evaluation of the set of sentences generated in this way is not possible. Nonetheless, it is worth remarking at this point that there are infinite candidates for such maps *S* from integers to sentences. For instance, here is a second putative map.

k	*S*(k)	Evaluation
-8	Snow is not white.	False
7	Snow is white.	True
-4	Socrates is not wise.	False
3	Socrates is wise.	True

-6	There is somebody who is undying.	False
5	All men are mortal.	True

Let *S* be any mapping that continues in this way indefinitely. That is for *S* we have

(A) $\forall k\ (k > 0 \rightarrow k$ is untrue) That is For all $k > 0$, $\sim S(k)$

(B) $\forall k\ (\sim S(k) \leftrightarrow S(-k\text{-}1))$.

Let *n* be a natural number ($n = 0, 1, 2 \ldots$), then we may construct a Yablo sequence of sentences:

(Y0) $S(0) \leftrightarrow \forall k.\ 0 < k \rightarrow \sim S(k)$

(Y1) $S(1) \leftrightarrow \forall k.\ 1 < k \rightarrow \sim S(k)$

(Y2) $S(2) \leftrightarrow \forall k.\ 2 < k \rightarrow \sim S(k)$

…

(Y*n*) $S(n) \leftrightarrow \forall k.\ n < k \rightarrow \sim S(k)$

(Y*n* + 1) $S(n + 1) \leftrightarrow \forall k.\ n + 1 < k \rightarrow \sim S(k)$

In the following formal demonstration of the antinomy, we will refer to (AR) when a line in the inference is a consequence of elementary arithmetic. Then the antinomy arises as follows:[1]

*(1) $S(n)$ suppose $S(n)$ is true

*(2) $\forall k.\ n < k \rightarrow \sim S(k)$ (Y*n*)

*(3) $n < n + 1 \rightarrow \sim S(n + 1)$ (2)

*(4) $n < n + 1$ (AR)

[1] If *m* is the last member of the sequence, then '$S(m)\ \&\ \forall k.k < m \rightarrow \sim S(k)$' is true, so the antinomy is eliminated. If $m = \omega_0$ (the smallest infinite ordinal) and $\forall x\ (S(x) \rightarrow (x$ – natural number or $x = m))$ then the sequence can be infinite. See: http://www.andrasek.hu/ferenc For the sake of simplicity we do not give formal definitions of '<' and '$n + 1$' but these would be possible.

*(5)	$\sim S(n + 1)$	(3) (4)
*(6)	$(\forall k.\ n < k \rightarrow \sim S(k)) \rightarrow (\forall k.\ n + 1 < k \rightarrow \sim S(k))$	(AR)
*(7)	$\forall k.\ n + 1 < k \rightarrow \sim S(k)$	(2) (6)
*(8)	$S(n + 1)$	(7) (Yn+1)
(9)	$S(n) \rightarrow (\sim S(n + 1)\ \&\ S(n + 1))$	(1) (5) (8)
(10)	$\sim S(n)$	(9)
(11)	$\forall n.\ \sim S(n)$	(10) n was arbitrary,
	hence by Universal Generalization	
(12)	$\forall k.\ n < k \rightarrow \sim S(k)$	(11) (AR)
(13)	$S(n)$	(12) (Yn)
(14)	$\sim S(n)\ \&\ S(n)$	(10) (13)

So one can produce the antinomy using more ontologically and formally parsimonious language. On the other hand the derivation is equivalent to Yablo's original inference. What is the consequence of this derivation, what kind of existence is refuted by the antinomy at (14)?

The antinomy demonstrates as a proof by contradiction that the evaluation does not exist.

Let us clarify this. It is clear that Yablo's infinite formula series can be generated by a Turing machine. The point is that only such a series of formulae exists but not true or false evaluations of those formulae as sentences. It is very important to distinguish the two categories: sentences and formulae. The map *S* exists, but the consistent evaluation of S(n), which is a series of formulae, is impossible. Alternatively, if we read S^* as the composite map of *S* followed by the evaluation, which is called an "evaluation map", then the non-existence of this S^* is demonstrated by this antinomy.

The nature of the "paradox" may be further demonstrated as follows. Because we restricted ourselves to integer variables we can translate the inference to arithmetic language.

Let f be an arithmetic function and k be an integer.

Let ∇ = XOR= eXclusive OR

The following results of arithmetic are also presupposed.

Even (0)

$\forall k$: Even (k) ∇ Odd (k),

$\forall k$: Even (k) $\leftrightarrow$ Odd (k + 1)

Let f be defined as follows.

Odd = $f(k)$ iff $S(k)$, Even = $f(k)$ iff ~$S(k)$

(Y0) Odd = $f(0) \leftrightarrow \forall k.\ 0 < k \rightarrow$ Even = $f(k)$

(Y1) Odd = $f(1) \leftrightarrow \forall k.\ 1 < k \rightarrow$ Even = $f(k)$

(Y2) Odd = $f(2) \leftrightarrow \forall k.\ 2 < k \rightarrow$ Even = $f(k)$

…

(Yn) Odd = $f(n) \leftrightarrow \forall k.\ n < k \rightarrow$ Even = $f(k)$

(Yn+1) Odd = $f(n + 1) \leftrightarrow \forall k.\ n + 1 < k \rightarrow$ Even = $f(k)$

The antinomy arise again more clearly as follows.

*(1)	Odd = $f(n)$	uppose value of f is odd for n
*(2)	$\forall k.\ n < k \rightarrow$ Even = $f(k)$	(Yn)
*(3)	$n < n + 1 \rightarrow$ Even = $f(n + 1)$	(2)
*(4)	$n < n$+1	(AR)
*(5)	Even = $f(n + 1)$	(3) (4)
*(6)	($\forall k.\ n < k \rightarrow$ Even = $f(k)$) $\rightarrow$ ($\forall k.\ n + 1 < k \rightarrow$ Even = $f(k)$)	(AR)
*(7)	$\forall k.\ n + 1 < k \rightarrow$ Even = $f(k)$	(2) (6)
*(8)	Odd = $f(n + 1)$	(7) (Yn + 1)
(9)	Odd = $f(n) \rightarrow$. Even = $f(n + 1)$ & Odd = $f(n + 1)$	(1) (5) (8)
(10)	Even = $f(n)$	(9)
(11)	$\forall n$. Even = $f(n)$	(10) n was arbitrary, hence by Universal Generalization

(12)	$\forall k.\ n < k \rightarrow \text{Even} = f(k)$	(11) (AR)
(13)	$\text{Odd} = f(n)$	(Yn)
(14)	$\text{Even} = f(n)\ \&\ \text{Odd} = f(n)$	(10) (13)

The question arises again, what kind of existence is refuted by (14)?
Yablo's $S(n)$ sequence is a consequence of a second-order logical formula.

(Ya) $\exists s \forall n$: n is a natural number $\rightarrow$ ($s(n) \leftrightarrow \forall k\ (n < k \rightarrow \sim s(k))$)

(In this formula s is a second order variable ranging over functions; the function S that we introduced at the beginning, a one-one injective map from a subset of integers to sentences, is one possible value of s.) This second-order formula is equivalent to the arithmetical statement

(Yb) $\exists f \forall n$: n is a natural number $\rightarrow$ ($\text{Odd} = f(n) \leftrightarrow \forall k\ (n < k \rightarrow \text{Even} = f(k))$)

The antinomy that is the consequence of this second-order formula refutes the existence as a proof by contradiction of such an f function and hence the existence of an infinite $S(n)$ sequence. The contradiction demonstrates that the infinite sequence of sentences that fulfils the stated conditions does not exist. This is what Yablo's antinomy proves.

References

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