Can we get in heaven with a hat on?

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At the entrance to heaven, there is a long queue; everyone has a number to avoid quarrels. Saint Peter will only open the gate if the following rule is fulfilled:

(i) *Anyone has a hat on his head if and only if no one behind him has a hat on his head.*

I will prove that Saint Peter need not bother to open the gate if there is an infinite queue at the gate, and someone—who is capable of such miracles—does not stand there last after an infinite number of people waiting, saying: "I am ω, the smallest transfinite ordinal number". (Zermelo-Fraenkel set theory miraculously has such a number.)

What if there were only one person at the gate? This is how a man can reason. If there were a hatted man behind me, then I would not put my hat on; but because there is no one standing, there is no hatted man behind me. Therefore, the other case applies, so I would have to put my hat on.
What if there were two people in line? The last person in line would still apply the same reasoning, so he would have a hat on. In this case, the person in front of him would be left without a hat. In the case of three people, the first two would not have hats on their heads, except for the last one. All this can be represented in this way:
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Is the following case possible?
$\bigcirc \otimes \otimes$

No, because the second one could only have a hat on his head if no one behind him had a hat; but he does, so he cannot have one on his head either. Is the following case possible?
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Let us apply the rule. The first man is not wearing a hat because the man behind him is. The second man has a hat, and the man behind him does not. The rule is fulfilled because there is no hatted man behind the second person. The third man has no hat. This breaks the rule because there is no hatted man behind him, so the third person must put on the hat. However, the moment the third one puts on a hat, the second one must remove his hat because there is now a hatted man behind him. Therefore, this case is not possible either. Let the number sign '1' denote the hatted and '0' the hatless. It can then be seen that only such a sequence satisfies the rule: $\langle 000 \ldots 0001 \rangle$, that is, only the last one can have a hat on. The question, however, is whether there is a last member of the series to base an evaluation on.

What is the general mathematical formulation of this problem? The question is whether there exists a function f that satisfies the following condition:

(1) $\forall n \forall k : n, k$ is a natural number $\rightarrow (1 = f(n) \leftrightarrow (n < k \rightarrow 0 = f(k)))$

I can prove that no such function exists when the sequence is restricted to natural numbers. Suppose, on the contrary, that there is such a function f_o. Then n is either hatted or not. Take the first: hatted $= f_o(n)$. If n is hatted, then everyone behind it is hatless, yes, but then $n + 1$ immediately behind it can wear a hat. However, if he puts it on, then the one in front of him cannot have a hat; therefore, n must be hatless: no hatted $= f_o(n)$. There is a contradiction; therefore, we must reject our assumption, that is, n must be hatless. However, this is true for anyone; therefore, no one can have a hat. Then, anyone can wear a hat, which is again a contradiction. Because f_o was arbitrary, it follows that such a function does not exist.

This is what the Yablo paradox proves. The situation changes if the domain of interpretation contains the smallest transfinite ordinal number ω in addition to the natural numbers. In this case, the last member of the sequence, ω, is the first transfinite ordinal number greater than any natural number.

In this case, there is a function f_2 such that for every natural number, the function has a value of 0 (= not hatted); whereas for the first transfinite ordinal, the function has a value of 1 (= hatted); see Figure 1.

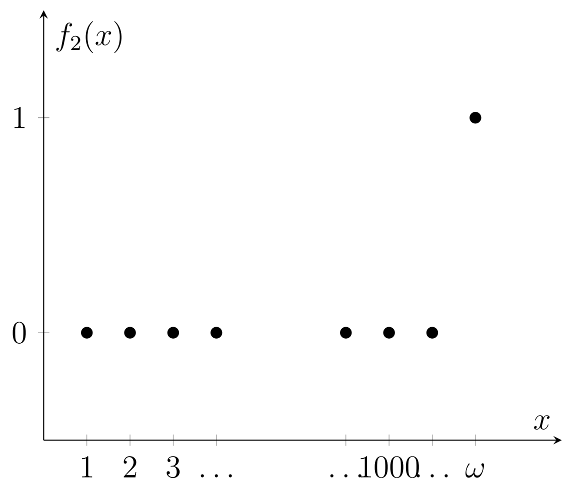

Figure 1: f_2 function

Swapping the variables k and n in '$<$' relation in Formula (1) yields the following formula:

(2) $\forall n \forall k : n, k$ is a natural number $\rightarrow (1 = f(n) \leftrightarrow (k < n \rightarrow 0 = f(k)))$

In natural language, this means the following rule:

(ii) *Anyone has a hat on his head if and only if no one before him has a hat on his head.*

This is how the first soul can reason: No one is ahead of me because I am the first in line. Consequently, there is no hatted man in front of me, which means I have to put my hat on. The second soul can think of the following reason: I see that the person in front of me has a hat on his head, so I cannot have a hat on my head. Finally, the third can think of the following reason: There is no hat on the head of the person in front of me, and because everyone follows the rule, it follows that someone in front of me has a hat on their head. Therefore, I cannot have a hat on my head either; and this is how everyone behind him thinks. It can then be seen that only such a sequence satisfies the (ii) rule: $\langle 100 \ldots 000 \rangle$, that is, only the first one can have a hat on. In the previous case, existence of the series depends on whether there is a last member of the series. Here, it depends on whether there is a first member of the series to base an evaluation on. Figure 2 shows the function defined by Formula (2).

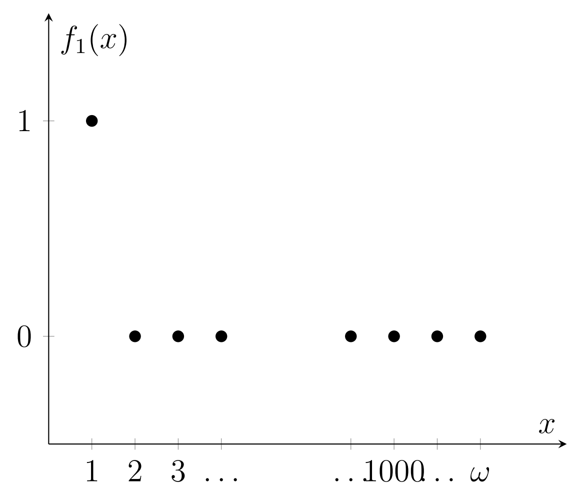

Figure 2: f_1 function

Restricting the domain of interpretation to natural numbers, Formula (2) defines an infinite series, whereas Formula (1) does not. The essential question is: Why does the second rule define a series and the first rule does not?

The "hatted" example was inspired by the papers of Graham Priest and Roy A. Sorensen.