From Zeno to Einstein

Extended version

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Abstract

Some experimental theories of quantum gravity, such as loop quantum gravity, propose a discrete or "quantized" structure for space-time at very small scales. These theories hypothesize that space-time is fundamentally made up of discrete units or "atoms" of space, in a similar way to how matter is fundamentally made up of discrete particles. In the context of space-time, the term "atomic structure" is used metaphorically to suggest a discrete or granular nature at extremely small scales. In Einstein's special theory of relativity, there is a maximum limit of speed, beyond which no point of mass in an inertial reference frame can travel. In the following, I will demonstrate that in a given inertial reference frame, in addition to the existence of an upper limit velocity of the motion of a point of mass, there are also logical reasons to think that space and time have an atomic structure. The basic idea of this argument was suggested by Zeno's well-known aporias.

Keywords: philosophy of physics, special theory of relativity, space-time, discrete space-time, Zeno's aporias

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Introduction

In Einstein's special theory of relativity, it is asserted that there is a maximum limit of speed, beyond which no point of mass in an inertial reference frame can travel. In the following, I will demonstrate in addition that in an inertial reference frame, there are also *logical reasons* to think that space and time have an atomic structure. The basic idea of the argument was suggested by Zeno's well-known aporias.[1]

Premises

Suppose that mass point m passes along a straight line from point A to point B, beginning at time t_1 and arriving at time t_2.

(1) We assume that the distance travelled and the associated time are both discrete and non-continuous, meaning nothing more than that there is an isomorphic order between the domain of space and time atoms and the domain of integers ordered by magnitude. In a given inertial frame, for any two instants t_1 and t_2, either t_1 is earlier than t_2, or t_2 is earlier than t_1, or t_1 is the same as t_2. In this way, we can represent both space and time coordinates by a series of integers. Then, assumptions (1.1) and (1.2) follow:

(1.1) There are only a finite number of points between points A and B, and for each of these, the previous and following points are clearly defined. For any point x point, $x + 1$ is the next point and $x - 1$ is the preceding one.

(1.2) The points of the route through which mass point m passes are arranged in such a way that we can decide determine of any two different points which is closer to the endpoint, namely to B.

(2) A point is somewhere at all times, that is, it has a place at every moment, and that place is a point on a line.

(3) If the point is somewhere, it is so at a time.

(4) The point m may move from a point to the next or previous one. The places to which the point may move at the following time are here called adjacent places. The concept of an adjacent space can also be interpreted in two or more dimensional discrete spaces, as I discuss later.

(5) In the case of present movement, a point never moves backward in time, so during the movement of the point m it touches all places between A and B, omitting and skipping none.

The following assumption is valid only in a deterministic world:

(6) The point is in one and only one place at each time, and the given place is a deterministic characteristic of the point. In another framework, it is also conceivable that the location of a point

[1] The first version of my thoughts appeared (in Hungarian) as part of a discussion paper in Hungarian Physical Review 2005/9. My source for the Zeno's aporias is Ruzsa (1966).

is interpretable only as a probability, i.e., the location of is a function of a statistical distribution. Let us suppose the following:

(7) A point is continuously advancing, so that if time x is later than y, then the position of the point at x is closer to B than the position at an earlier time y. Thus, it follows that the point moves continuously the range between t_1 and t_2,including t_1 and t_2, as it travels. In continuous motion, if the two times are not the same, the two corresponding points of the point are also different.

Conclusion

Thus, the point can only travel at a single speed, which is the maximum speed as determined by the shortest length and duration.

Proof

Suppose that at one time atom, a point travels two or more space atoms. Because we assumed that the point omits no space and is always somewhere, we cannot know on a discrete time scale where the point is if it is travelling across several spatial atoms at one time. It must be somewhere, and we assumed it could not be in multiple places at one time. This contradicts its necessity to travel across no more than one space atom at one time. In contrast, if the point travels more slowly than the maximum speed, then some two adjacent times show the point at the same place, which in turn violates the condition of continuous progress. If continuous progress is discarded, slower velocities can be matched by the intermittent advancement of the point and interspersed with times where it remains in place for longer or shorter periods of time, depending on the velocity. Thus, the point may travel more slowly than the maximum speed in such a special intermittent manner. (See figure 1 below.)

Proof in formal logic language

Notations:

$S(x) :=$ x is a location

$S(x) \wedge S(y) \wedge x < y :=$ y is further from the origo than x

$T(x) :=$ x is a time atom

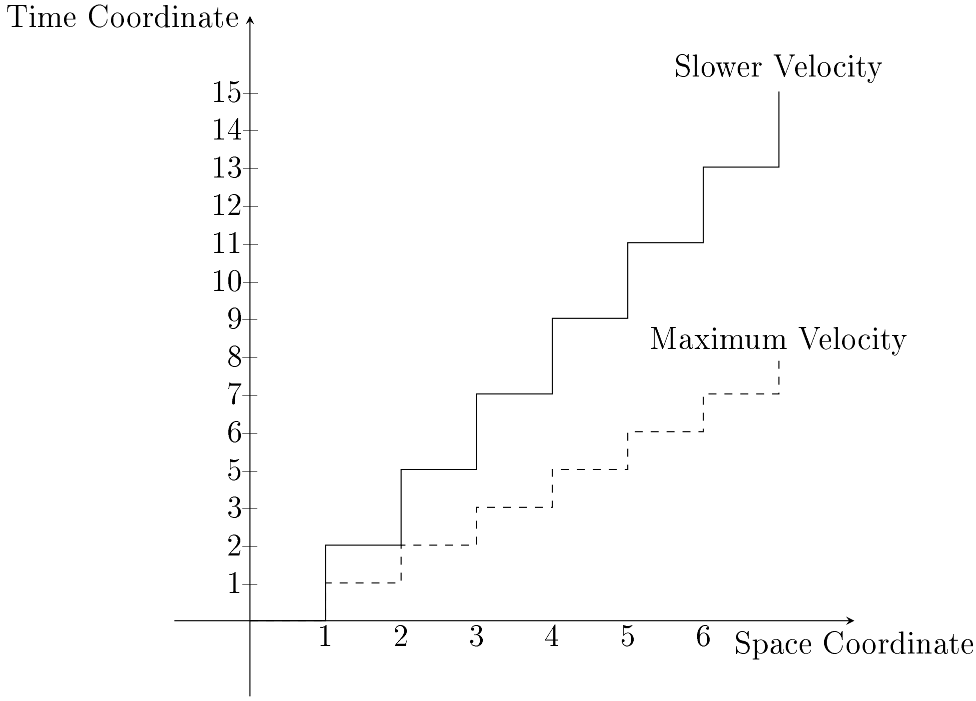

Figure 1: World lines in atomic space-time

$T(x) \wedge T(y) \wedge x < y :=$ y later than x

s_m is the time-distance function of m mass point

$y = s_m(x) :=$ distance of m mass point is y at x time

Premises:

(1) $\forall x \forall y \forall z (\neg x < x \wedge (x < y \rightarrow \neg y < x) \wedge ((x < y \wedge y < z) \rightarrow x < z))$

The relation $<$ is asymmetric and transitive (strict ordering).

(2) $S(A) \wedge S(B) \wedge \neg\exists x(S(x) \wedge x < A) \wedge \neg\exists x(S(x) \wedge B < x)$

The distance has a start (A) and end point (B).

(3) $\forall x \forall y((T(x) \wedge T(y)) \rightarrow (x < y \vee y < x \vee x = y))$

(4) $\forall x \forall y((S(x) \wedge S(y)) \rightarrow (x < y \vee y < x \vee x = y))$

Any two time or place atoms are comparable.

(5) $\forall x(x-1 < x \wedge x < x+1) \wedge \forall x \neg\exists y(x < y \wedge y < x+1) \wedge \forall x \neg\exists y(x-1 < y \wedge y < x) \wedge \forall x(x \neq x-1) \wedge \forall x(x \neq x+1))$

Space and time have an atomic structure.

(6) $\forall y(S(y) \to \exists x(y = s_m(x) \wedge T(x))) \wedge \forall x(T(x) \to \exists y(y = s_m(x) \wedge S(y)))$

(7) $\forall x \forall y \forall z((S(x) \wedge S(y) \wedge S(z) \wedge x < y \wedge y < z) \to$
$\exists u \exists v \exists w(T(u) \wedge T(v) \wedge T(w) \wedge x = s_m(u) \wedge y = s_m(v) \wedge z = s_m(w) \wedge u < v \wedge v < w))$

The moving point does not miss or skip places.

(8) $\forall x \forall y((T(x) \wedge T(y) \wedge x < y) \to s_m(x) < s_m(y))$

A point is continuously advancing.

Deductions:

Suppose that m moves more than one distance atom in a period:

(9)	$y = s_m(x) \wedge z = s_m(x+1) \wedge y + 1 < z$	
	This is impossible because it contradicts (5)(7).	
(9.1)	$a = s_m(x) \wedge b = s_m(x+1) \wedge a + 1 < b$	(9)
(9.2)	$a < a + 1 \wedge a + 1 < b$	(5)(9.1)
(9.3)	$a = s_m(x) \wedge b = s_m(x+1)$	(9.1)
(9.4)	$\exists v(a = s_m(x) \wedge a + 1 = s_m(v) \wedge b = s_m(x+1))$	(7)(9.3)(9.2)
(9.5)	$\exists v(x < v \wedge v < x + 1)$	(7) (9.4)
(9.6)	$\neg \exists v(x < v \wedge v < x + 1)$	(5)
(9.7)	$\neg \exists v(x < v \wedge v < x + 1) \wedge \exists v(x < v \wedge v < x + 1)$	(9.5) (9.6)

Suppose that m passes less than one atom distance.

(10)	$y = s_m(x) \wedge z = s_m(x+1) \wedge y < z \wedge z < y + 1$	
	This is impossible because it contradicts (5).	
(10.1)	$a = s_m(x) \wedge b = s_m(x+1) \wedge a < b \wedge b < a + 1$	(10)
(10.2)	$\neg \exists y(a < y \wedge y < a + 1)$	(5)
(10.3)	$\exists y(a < y \wedge y < a + 1)$	(10.1)
(10.4)	$\neg \exists y(a < y \wedge y < a + 1) \wedge \exists y(a < y \wedge y < a + 1)$	(10.2)(10.3)

Suppose that m does not move in a range of time.

(11) $y = s_m(x) \wedge y = s_m(x+1)$

This is impossible because it contradicts (8).

(11.1)	$a = s_m(1) \wedge a = s_m(2)$	(11)
(11.2)	$s_m(1) < s_m(2)$	(8)(11.1)
(11.3)	$a < a$	(11.1)(11.2)
(11.4)	$\neg a < a$	(1)
(11.5)	$\neg a < a \wedge a < a$	(11.3)(11.4)

The remaining possibility is that m mass point move continuously with maximum velocity.

(12) $\forall x \forall y((T(x) \wedge S(y)) \rightarrow (y = s_m(x) \rightarrow y + 1 = s_m(x+1)))$

As I noted earlier, if continuous progress is discarded, slower velocities can be matched by the intermittent advancement of the point and interspersed with times where it remains in place for longer or shorter periods of time, depending on the velocity. Thus, the point may travel more slowly than the maximum speed in such a special intermittent manner.

Open questions

This intermittent progress is peculiar but not contradictory, but the atomic structure of space-time raises a number of additional questions.

In the figure, I plot coordinates using a regular straight line. However, if the distance is indeed made up of discrete atomistic places, the regular arrangement of discrete space atoms is not in evidence. Only condition (1) determines the logic of discrete spatial atoms. Viewed from a continuous world, the spatial atoms of the discrete world may also be located on a wave—they are not limited to being a straight line. It follows that a square grid representing one-dimensional space-time does not necessarily have to exhibit a regular geometric shape when viewed from a continuous world. If we draw a square grid of one-dimensional space-time on a rubber sheet that we can stretch and shrink the rubber sheet, it still the logical requirements are still met. What is the essence of these logical requirements? Namely, that you cannot skip grid points in space-time. This is the reason why I also provide a stair graph for maximum speed and not simply a line.[2] The

[2] I take the consideration of grid points from a 2006 lecture by Tamás Sándor Bíró at the Skeptic Conference.

significance of this can be better understood when we think in two-dimensional space.

Why do we consider that the distance between two points is represented by a straight line and not a curved line? This is because we believe distance is the shortest path between the two points, and the piece of rope corresponding to a straight line is shorter than the rope corresponding to a curved line. For simplicity's sake, let us stick to regular layouts. Let the twelve atomic locations given in a two-dimensional discrete world be in the following arrangement (See table 1 below.):

A	E	I
B	F	J
C	G	K
D	H	L

Table 1: Two dimensional finite space

How far is A from K? It depends on how we move forward. In point (3), I declared that in one-dimensional space, the motion can always take place only at the adjacent location at consecutive, discrete times. In one-dimensional space, the concept of adjacent space is obvious, but in two-dimensional or multidimensional space, this is no longer the case, and we can choose between several options. If we move obliquely, then touching F, the distance between two atoms between A and K is the shortest path. If we can only move left or right and up or down, then the distance is four square spaces.

How far apart are points A and D? Our intuition suggests that they are at three atoms' distance. Is this the shortest distance between A and D? The answer to this now also depends on how we expect to proceed. If we are allowed to move in any direction between the grid points, we can connect D to E. Point A is next to point E, and so there are only two atomic distances between A and D. If the shortest path between two points is the distance, and nothing is forbidden by this path, we must accept that the distance between A and D is three. However, this contradicts the previous stipulation, namely, that points cannot be skipped or avoided in one-dimensional travel. However, we have just done this, so we rule out this possibility. The question is whether movement such as from A to F, is permissible: can we move obliquely? (A similar question can be asked with respect to three-dimensional space.)

How far apart are A and L? If we allow oblique travel but prohibit jumping, there are three different routes of three atomic distances A and $L: A-B-G-L, A-F-K-L, A-F-G-L$. The understanding of a line connecting two points is therefore unclear. If we stipulate that the position of a point can only change to the right or left and up or at any instance—so there is no oblique motion—the line remains ambiguous, but the shortest distance between A and L becomes five atoms. However, the Pythagorean Theorem is not valid in any case, nor does it help to divide distances by many points.[3]

A continuous space-time would come in handy in avoiding these dilemmas. Perhaps a continuous space-time model can be mapped to a probabilistic atomic space-time model, where random information replaces the surplus information of the continuous quantities (in real numbers). This would be a solution to the problem of atomic space-time.

Epilogue

At present, there are no widely embraced physical theories positing an atomic structure for space-time. For classical physics, the prevailing view is that space and time are continuous and infinitely divisible. This notion of continuous space and time is also ingrained in Einstein's general relativity, in which gravity is depicted as the curvature of space-time. However, it is noteworthy that certain experimental theories in the realm of quantum gravity, such as loop quantum gravity, propose a discrete nature of space-time at extremely small scales. These theories hypothesize that space-time is fundamentally composed of discrete units or "atoms" of space, similar to the way in which matter consists of discrete particles. The proposed limitation on the time interval in these theories is approximately 10^{-43} seconds, known as the Planck time. The corresponding Planck distance, representing the distance that light travels in one unit of Planck time, is estimated to be around 10^{-35} meters.

[3] McDaniel (2007).

Bibliography

András, Ferenc.(2005). "Notes on an interpretation of special relativity". *Physical Review*:2005/9 (in Hungarian)

Crouse, David and Joseph D. Skufca. (2018). "On the Nature of Discrete Space-Time: Part 1: The distance formula, relativistic time dilation and length contraction in discrete space-time." *arXiv: General Physics.*

Huggett, Nick, ed. (1999). *Space from Zeno to Einstein.* Cambridge, MA:Bradford.

Maudlin, Tim. (2012). *Philosophy of Physics: Space and Time.* Princeton University Press. https://doi.org/10.2307/j.ctvc77bdv.

McDaniel, Kris. (2007). "Distance and Discrete Space." *Synthese* 155 (1): 157–62.

Ruzsa, Imre and János Urbán. (1966). *A matematika néhány filozófiai problémájáról/Matematikai logika* (*Some philosophical problems in mathematics/Mathematical logic*). Budapest: Tankönyvkiadó.