

A few comments on Roy T. Cook's "Curry, Yablo and duality" paper
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In his thought-provoking paper, Roy T. Cook investigates two infinite sequences of formulas.¹ He claims something about the formulae and gives us proof of them. My reply consists of two parts. The first part (1–3) is rather polemical, but in the second part (4), I will confirm in an elegant way what Cook claims is true, hopefully to the reader's satisfaction.

1. Cook formulates Tarski's T schema this way:

" $T(\langle \Lambda \rangle) \leftrightarrow \Lambda$ " p. 612

He noticed: "Here ' $\langle \dots \rangle$ ' is any appropriate name-forming device, such as Gödel coding. ... where T is a truth predicate ... In addition, here and throughout, I shall assume that negation and the truth predicate commute, so that: $\sim T(\langle \Phi \rangle)$ and: $T \langle \sim \Phi \rangle$ are inter-derivable (in other words, I assume that falsity is equivalent to failure to be true)." His formulation of the T schema above is acceptable as a basic idea, but not when we are talking about paradoxes. In this context a philosopher must be very rigorous in applying formal languages.

It is clear that Λ above is a sentence variable or a sentence schema and not a sentence constant like: "This statement is false." Therefore I understand that ' $(\lambda \Lambda) \langle \Lambda \rangle$ ' is a function, a map from sentences to sentence names. The author does not specify different maps: ' $\langle \dots \rangle_1, \langle \dots \rangle_2, \langle \dots \rangle_3$ '; so $\langle \dots \rangle$ is a name-forming device in general, and therefore Cook's T schema is presumably this: $\forall \langle \dots \rangle \forall s T(\langle s \rangle) \leftrightarrow s$, which is suspicious. Let us explain why.

Tarski called the ' $(\lambda x) \langle x \rangle$ ' function a 'quotation function', distinguishing the usage of the quotation function from the normal usage of quotation marks. We know from Tarski that a quotation function in itself is a very risky device; it can generate antinomy alone

in closed languages. Tarski carefully investigated the next variant of the T schema in his famous article “The concept of truth in formalized languages”:

$\forall \langle \dots \rangle \forall x (\text{True}(x) \leftrightarrow \exists s (x = \langle s \rangle \& s))$ which is logically equivalent to Cook’s T schema.

After all, Tarski rejected the above formulation of the T schema, and he had a good reason for doing so. Tarski only sketched the argument; here is a detailed explanation following Imre Ruzsa. Let p and q be sentence variables, and A and B sentence parameters; suppose (β') .

$$(\beta') \forall p \forall q (\langle p \rangle = \langle q \rangle \rightarrow (p \leftrightarrow q))$$

There are no restrictions for the naming-device functor, and p can be an arbitrary sentence, e.g. a sentence containing a quantifier and a naming-device functor itself, so it is allowable to apply our naming device in this way:

$$(\alpha') \langle A \rangle = \langle \forall p (\langle A \rangle = \langle p \rangle \rightarrow \sim p) \rangle$$

$$(\alpha'') A \leftrightarrow \forall p ((A \leftrightarrow p) \rightarrow \sim p) \quad (\beta')$$

$$(\alpha''') A \leftrightarrow ((A \leftrightarrow B) \rightarrow \sim B)$$

$$(\alpha'''') A \leftrightarrow \sim A$$

Yes, we derived a contradiction without any assistance of a Liar or Yablo paradox by just applying our naming device in a closed language. It is similar to Cook’s formal language in question, so he applies an inconsistent language generating paradoxes, which is rather depressing.

2. How to formulate the Liar sentence? This is Cook’s answer:

$$\Lambda: \sim T(\langle \Lambda \rangle)$$

What does the colon mean in the above? In some contexts, a colon introduces a list or a set, referring to an individual term or an object, or defines a predicate, sometimes introducing the logical consequence, and there are many other usages. In our case, Λ is a sentence – not a name – and ' $\sim T(\langle \Lambda \rangle)$ ' is also a sentence, so this colon means ‘if and only if’, or the introduction of a definition. Later, however, Cook introduces a definition using the ‘ $=_{df}$ ’ sign: “If we define negation in terms of conditional a primitive absurdity

constant ‘ \perp ’: $\sim\psi =_{df} \psi \rightarrow \perp$ ” (p. 613) we have no other choice than the first option. The letter ‘ Λ ’ is a variable or schema name in Cook’s formulation of T schema, but in the formulation of the Liar paradox it should be a constant, not a variable.

3. There are two typos on p. 618:

(f) a universal quantifier was missing: $(\forall m > j + 1) (T(\langle Sm \rangle) \rightarrow \Phi)$

(g) $T(\langle S_{j+1} \rangle)$ and not S_{j+2}

4. Now I demonstrate an alternate proof of Cook’s main theorems by applying an arithmetic translation. Let n and m be natural number variables, S a unary predicate, and α a unary predicate variable.

Cook’s first claim (Dual Yablurry), without semantic predicates and replacing colons with ‘ \leftrightarrow ’ symbols is:

(1) $S(1) \leftrightarrow (\forall m (m > 1 \rightarrow S(m)) \rightarrow \Phi)$

(2) $S(2) \leftrightarrow (\forall m (m > 2 \rightarrow S(m)) \rightarrow \Phi)$

....

(n) $S(n) \leftrightarrow (\forall m (m > n \rightarrow S(m)) \rightarrow \Phi)$

....

The sentence series above claims that Φ follows from:

$\exists \alpha \forall n (\alpha(n) \leftrightarrow (\forall m (m > n \rightarrow \alpha(m)) \rightarrow \Phi))$ second order logic formula.

The next inference has only five lines.

* (1) $\exists \alpha \forall n (\alpha(n) \leftrightarrow (\forall m (m > n \rightarrow \alpha(m)) \rightarrow \Phi))$

* (2) $\forall n (2 * n + 1 \text{ is odd} \leftrightarrow (\forall m (m > n \rightarrow 2 * m + 1 \text{ is odd}) \rightarrow \Phi))$ (1)

I substituted α with the ‘ $2 * \textcircled{1} + 1$ is odd’ predicate, but it also works with the ‘ $\textcircled{1} = \textcircled{1}$ ’ predicate.

* (3) $(\text{True} \leftrightarrow (\text{True} \rightarrow \Phi))$ (2) from elementary arithmetic

* (4) Φ (3)

(5) If $\exists \alpha \forall n (\alpha(n) \leftrightarrow (\forall m (m > n \rightarrow \alpha(m)) \rightarrow \Phi))$ then Φ (4)

Second claim (Yablurry):

Φ follows from: $\exists \alpha \forall n (\alpha(n) \leftrightarrow (\forall m (m > n \rightarrow (\alpha(m) \rightarrow \Phi))))$

It is easy to prove the second formula in a similar way.

* (1) $\exists \alpha \forall n (\alpha(n) \leftrightarrow (\forall m (m > n \rightarrow (\alpha(m) \rightarrow \Phi))))$

* (2) $\forall n (2 * n + 1 \text{ is odd} \leftrightarrow (\forall m (m > n \rightarrow (2 * m + 1 \text{ is odd} \rightarrow \Phi))))$ (1)

* (3) $(\text{True} \leftrightarrow (\text{True} \rightarrow \Phi))$ (2) from elementary arithmetic

* (4) Φ (3)

(5) If $\exists \alpha \forall n (\alpha(n) \leftrightarrow (\forall m (m > n \rightarrow (\alpha(m) \rightarrow \Phi))))$ then Φ (4)

So I verified both Cook's claims, and it is important to note that I used only elementary arithmetic theorems, without using either the omega-rule or T schema.

¹ Roy T. Cook: Curry, Yablo and duality. Analysis Vol 69 | Number 4 | October 2009